# THE SOLUTIONS OF THE DIFFERENTIAL EQUATION

$$v''' + \lambda^2 zv' + 3\mu\lambda^2 v = 0$$

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1. **Introduction.** A differential equation

$$\frac{d^n w}{dz^n} + \lambda p_1(z, \lambda) \frac{d^{n-1} w}{dz^{n-1}} + \cdots + \lambda^n p_n(z, \lambda) w = 0, \tag{1.1}$$

in which $\lambda$ is a complex parameter of large absolute value, and the coefficients are expressible (actually or asymptotically) by power series in $1/\lambda$, thus

$$p_j(z, \lambda) = \sum_{r=0}^{\infty} \frac{p_{j,r}(z)}{\lambda^r}, \qquad j = 1, 2, \cdots, n, \tag{1.2}$$

has associated with it the so-called *characteristic* or *auxiliary* algebraic equation

$$\chi^n + p_{1,0}(z)\chi^{n-1} + \cdots + p_{n,0}(z) = 0. \tag{1.3}$$

The structure of the solutions of the differential equation in a given region of the complex $z$-plane depends in an essential way upon the configuration of the roots of the equation (1.3) in that region. If the roots are distinct throughout the region, or if they are distinct except for coincidences that maintain identically over the region, the forms of the solutions, as they depend asymptotically upon $\lambda$, are determinable by well established theories that are quite complete [1].

That is not so if the region of $z$ is one within which a coincidence of roots of the equation (1.3) occurs at an isolated point. Such a point is called a *turning point* of the differential equation. In the presence of such a point the $z$-region is divided into parts in each of which the solutions have distinctive forms. Both these forms and the inter-relations between the forms in different sub-regions must be determined.

A method by which certain classes of differential equations (1.1) with turning points have been studied is that of "related" equations. This calls for the construction of a differential equation which is explicitly solvable, and which is identical with the given equation to the extent of all terms up to those in an arbitrarily prescribed power of $1/\lambda$. The construction of such a related equation must be based upon a suitable differential equation whose solutions are completely known, and which manifests the same configuration of auxiliary roots as does the given equation. When the coincidence of auxiliary roots is suitably simple and applies to only a single pair of them the role of this basic equation is assumed either by the Bessel equation or the equation of a certain confluent hypergeometric function. This accounts for the fact that the solutions of large

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