FINITELY GENERATED MODULES

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It is our purpose to give still another presentation of the structure theory of finitely generated modules over a principal ideal ring. The main distinction between this and previous treatments lies in a simple invariant characterization of free modules used as a tool in the proofs rather than obtained as a corollary of the invariant factor theory, and an emphasis on the geometric meaning of determinants as a measure of relative weight or volume. As a consequence, the uniqueness theorem appears as a sort of maximum-minimum formula. The references below contain alternative approaches: [1], pp. 19–20; [2], pp. 65–108; [3], pp. 48–53; [4], p. 85 ff.; and [5] §§108, 109.

Let $\mathfrak{o}$ denote an integral domain which is a principal ideal ring. To avoid endless repetition, we shall understand that the elements of $\mathfrak{o}$ appearing in multiplicative equations are fixed modulo units. The g.c.d. $(a_1, \cdots, a_n)$ of several elements is either the ideal they generate or a generator of that ideal, as is convenient.

By an $\mathfrak{o}$-module E we shall mean what is normally designated a finitely generated unitary $\mathfrak{o}$-module. A free one is one with a basis of free elements (or a free basis); the number of basal elements is called the rank, $\rho(E)$. If E is an $\mathfrak{o}$-module and $x_1, \cdots, x_n \in E$, then $F = \mathfrak{o}x_1 + \cdots + \mathfrak{o}x_n$ denotes the submodule generated by them. If $x_1, \cdots, x_n$ are free so that F is a free module and n is maximal, n is called the rank of E, $n = \rho(E)$. One easily sees, using linear equations, that each $\mathfrak{o}$-module has finite rank.

If $x \in E$, $\mathfrak{n}(x) = \{a \in \mathfrak{o} \mid ax = 0\}$ is the *annihilator* of x, an ideal in $\mathfrak{o}$. If $\mathfrak{n}(x) = 0$, x is free; otherwise x is called a *torsion element*. The totality T of torsion elements is a submodule of E, the *torsion module*.

A basic fact we shall assume, since its proof is readily available [4] is that each $\mathfrak{o}$-module E has the ascending chain condition (ACC) for submodules, a consequence of the ACC for ideals of $\mathfrak{o}$ and the finite generation of E.

Lemma 1. *Let E be an $\mathfrak{o}$-module, T its torsion module. Then E is free if and only if $T = 0$.*

Proof. If E is free then obviously $T = 0$. Conversely, assume $T = 0$ and set $n = \rho(E)$. Thus there exist free rank n submodules of E; let F be a maximal one. We must prove $F = E$ and begin by taking a free basis $x_1, \cdots, x_n$ of F. We set $F_r = \mathfrak{o}x_1 + \cdots + \mathfrak{o}x_r$, a free module of rank r for $r = 1, \cdots, n$ and shall show, by induction, that if $x \in E$, $c \neq 0$, and $cx \in F_r$, then $x \in F_r$. The passage from $r - 1$ to r also includes the case $r = 1$ so we need not treat that case separately. We have $cx = c_1x_1 + \cdots + c_rx_r$ with $c \neq 0$. Set $d = (c, c_r) = ac + bc_r$ and $y = bx + ax_r$. Then $cy = b(cx) + (ac)x_r = b(c_1x_1 + \cdots + c_rx_r) +$

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