A MAPPING THEOREM FOR METRIC SPACES

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1. **Introduction.** It is well known (see for example [2; 252, §66.2]) that if R is a compact metric space, and f a closed continuous map of R onto a topological space R^*, then $R^* = f(R)$ is also a metric space. Since Whyburn has proved [3; 73, Theorems 9, 10] that normality and strong (= perfect) separability are preserved by a closed continuous map f, provided the image space R^* is assumed locally separable, it follows (by Urysohn's metrization theorem) that in the theorem stated in the beginning the hypothesis "R is a compact metric space" can be replaced by "R is a separable metric space" (provided R^* is assumed locally separable). The question arises whether the "separability" restriction on R can also be eliminated; i.e., whether a closed continuous (locally separable) image R^* of *any* metric space R is a metric space.

In this note we show that the answer is in the affirmative for all closed continuous maps which are also open. The problem in the general case, however, remains unanswered.

In the proof of our result, it is interesting to note that we do not make use of any "metrization theorems" as in the case of the previously referred result. Rather we obtain explicitly a metric for the image space R^* by making use of the well-known Hausdorff distance function $d(X, Y)$ for closed sets of the original space R.

2. **Definitions and preliminary remarks.** As usual a map f is called closed (open) if it transforms every closed (open) set into a closed (open) set of the image.

In this paper we use the term *topological space* for any set R satisfying the three standard neighborhood axioms of Hausdorff. (Hence no separation postulates are to be assumed unless explicitly stated.)

A topological space R in which single points are closed will be called T_1. A topological space R satisfying the 1-st (2-nd) axiom of countability will be called *locally* (*strongly*) separable [2; 74, §21]. In a locally separable space R, if $\{G_n\}$ $(n = 1, 2, \cdots)$ is a countable fundamental family of open neighborhoods of a point p, then this can be replaced, whenever necessary, by the countable fundamental family $\{H_n\}$ where $H_n = \prod_{i=1}^{n} G_i$, which has the additional property of being a decreasing family.

A sequence of points $\{p_n\}$ in a topological space R is said to *converge to* p (in symbols, $p_n \to p$) if every open neighborhood of p contains all but a finite number of the p_n. It follows from the definition that if $p_n \to p$ and (p_{n_i}) is a subsequence of (p_n), then $p_{n_i} \to p$.