THE LAPLACIAN OF FOURIER TRANSFORMS

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1. Let $f(x)$ in class $C^{(2)}$ on every bounded domain in Euclidean k-space, $k \geq 2$, be the Fourier transform of a function $g(u)$ in L_1 on E_k. If $g(u)$ is sufficiently "nice", then it is not difficult to show that the Fourier transform of $\Delta f(x)$, where Δ is the Laplacian operator, is equal to $-(2\pi)^k \mid u \mid^2 g(-u)$. In general, however, the Fourier transform of $\Delta f(x)$ is not even defined. It is the purpose of this paper to show that nevertheless the spherical partial integrals of $\Delta f(x) \exp(i(x, u))$ do tend in some sense to the desired value. In particular it is shown here that they are Riesz summable of order δ (δ satisfying the condition of Bochner [1; Theorem 1]) as well as Abel summable (see Hardy [4; 135]) almost everywhere to the desired value.

We show here, also, that similar statements can be made if $g(u)$ in the above is assumed in L_2 on E_k instead of L_1, though a further condition has to be added in the case of Abel summability.

In this paper the notion of spherical (A, n) summability [4; 71] is introduced, and it is shown that the above results also go over for this method.

We close this paper with an application of the Riesz summability results to the uniqueness theory of multiple trigonometric integrals.

2. The notation x and u will denote points in Euclidean k-space, $k \geq 2$, $x = (x_1, \cdots, x_n)$ and $u = (u_1, \cdots, u_n)$; (x, u) will denote the scalar product $x_1u_1 + \cdots + x_nu_n$; and $\mid x \mid$ will equal $(x, x)^{\frac{1}{2}}$.

The open k-dimensional sphere with center x and radius r will be designated by $D_k(x, r)$ and the surface of the sphere by $C_k(x, r)$.

Following Bochner [1], we shall say that the trigonometric integral $\int_{E_k} c(u) \cdot \exp i(x, u)\, du$ is spherically convergent at the point x to the finite-value $L(x)$ if the spherical partial integrals of rank R converge to $L(x)$, that is if

$$I_R(x) = \int_{D_k(0,R)} e^{i(x,u)} c(u)\, du \rightarrow L(x) \tag{1}$$

as $R \rightarrow \infty$.

We shall say that the above integral is spherically Riesz summable of order $\alpha > 0$, that is spherically (C, α) summable, to $L(x)$ if

$$\sigma_R^{(\alpha)} = 2\alpha R^{-2\alpha} \int_0^R I_r(x)(R^2 - r^2)^{\alpha-1} r\, dr \rightarrow L(x) \tag{2}$$

as $R \rightarrow \infty$.

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