MULTIPLICATIVE SEMI-GROUPS OF CONTINUOUS FUNCTIONS ON A COMPACT SPACE

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1. **Introduction.** Let $C = C(X)$ be the Banach algebra of all real (or complex) continuous functions on a compact Hausdorff space X. A well-known result of Stone [5] states that the proper closed ideals of C are the subsets of the form $k(F) = \{f \in C \mid f(t) = 0, t \in F\}$ where F is a non-void closed set in X. For a dense sub-algebra A of C the sets $k(F) \cap A$ need not be all the proper closed ideals; precisely when this is so is at present quite obscure. In this case (and others) the status of the sets $k(F) \cap A$ becomes much clearer if we consider them as sub-semi-groups of A viewed as a multiplicative semi-group. We obtain a necessary and sufficient condition on A for these sets to be distinct and to be all the proper closed sub-semi-groups of a certain type (all the proper closed S-ideals) and in this way a significant extension of Stone's theorem is achieved.

Let B be a multiplicative semi-group in C which, for some $K > 0$, contains in its closure for each pair F_1, F_2 of disjoint closed sets in X a function u, $\| u \| \leq K$, $u(t) = 0$, $t \in F_1$ and $u(t) = 1$, $t \in F_2$. It is shown that the sets $k(F) \cap B$ where F is closed and non-empty are distinct and are the proper closed S-ideals of B if, and only if for each pair F_1, F_2 and each $\epsilon > 0$ there exists $v \in B$ such that $v(t) = 0$, $t \in F_1$ and $| v(t) - 1 | < \epsilon$, $t \in F_2$. (No requirement is put on $\| v \|$.)

As suggested by the referee, the appropriate abstract setting for this investigation is C given as an abstract semi-group and as a topological space. The properties of the semi-group B mentioned above (and all others employed) are intrinsic in that they are definable entirely in terms of C so given. Thus we determine here relations between intrinsically defined subsets of C and the space X. In §4 we take up the problem of recovering X from a semi-group B in C. This is accomplished for suitable B by topologizing the set of maximal proper closed S-ideals of B.

A study of C as a multiplicative semi-group, made by Milgram [3], is essential background for this paper. We are also indebted to it for suggesting some devices used below.

2. **Notation and preliminaries.** Let $C(X)$ be the algebra of all real (or all complex) continuous functions on a compact Hausdorff space X with the usual norm $\| f \| = \sup | f(t) |$, $t \in X$. Suppose however that $C(X)$ is presented as a (multiplicative) semi-group and as a topological space. Thus viewed, $C(X)$ will, in this section, be denoted by C. Note that in C we are not given the

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