CONTINUATION OF BIHARMONIC FUNCTIONS BY REFLECTION

By R. J. Duffin

1. **Introduction.** Let the function $u(x, y, z)$ be harmonic in the hemisphere $x^2 + y^2 + z^2 < a^2$, $x > 0$ and satisfy the boundary condition $u \to 0$ as $x \to 0$. Then u may be continued so as to be harmonic in the whole sphere by the relation

$$u(-x, y, z) = -u(x, y, z). \tag{1}$$

This is the well-known Schwartz reflection principle. In this paper analogous continuation formulae for biharmonic functions are analyzed.

Let $w(x, y, z)$ be biharmonic in the hemisphere and satisfy the boundary condition $w/x \to 0$ as $x \to 0$. Then it is shown that w may be continued so as to be biharmonic in the whole sphere by the relation

$$w(-x, y, z) = -w + 2x\, \partial w/\partial x - x^2 \Delta w. \tag{2}$$

Taking w to be a function independent of z, it is apparent that the same formula gives a continuation in the analogous two-dimensional situation.

In the two-dimensional case relation (2) was found by Poritsky [8] and applied to problems of planar elasticity. This paper extends Poritsky's work in the following ways: (a) It is not necessary to assume *a priori* that w satisfies the biharmonic equation on the boundary. (b) Both two and three dimensions are treated. (c) Various boundary conditions are considered. (d) Spherical boundaries are treated. (e) Continuation formulae are found for viscous flow.

Other boundary conditions are considered which are important in the theory of plates. The deflection $w(x, y)$ of a thin flat plate satisfies $\Delta\Delta w = f$ where f is the pressure on the surface. At a part of the boundary where the plate is clamped, the boundary conditions are

$$w = 0 \qquad \text{and} \qquad \partial w/\partial n = 0. \tag{3}$$

Here n denotes the exterior normal. It is clear that these two conditions are essentially equivalent to the single condition $w/x \to 0$.

At a portion of the boundary where the plate is simply supported (hinged), the boundary conditions are

$$w = 0 \qquad \text{and} \qquad (1 - \mu)\, \partial^2 w/\partial n^2 + \mu \Delta w = 0. \tag{4}$$

Here the constant μ is Poisson's ratio and is usually assumed to be in the range $0 \leq \mu < 1$. These conditions are shown to imply the simple continuation formula (1).

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