MULTI-HOMOTOPY

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1. **Introduction.** One of the many definitions for continuity of a multi-valued function is closely enough related to continuity of single valued functions to allow one to generalize a considerable portion of the homotopy theory. In order to state this definition the following notation is introduced. Multi-valued functions will be denoted by F, G, H and single valued functions by f, g, h. Let $(D, *)$ be a directed set. Let X, Y, Z denote topological spaces and x, y, z, elements of X, Y, and Z, respectively. The closed unit interval of real numbers is denoted by I. A point $y_o \in Y$ is said to be in the *cofinal limit* of a sequence of sets $\{Y_d\}$, indexed by a directed set D, if whenever V is an open set containing y_o there is a cofinal subset C contained in D such that $c \in C$ implies that $V \cap Y_c \neq 0$. Similarly, y_o is an element of the *residual limit of* $\{Y_d\}$ if there is a residual subset R contained in D such that $r \in R$ implies that $V \cap Y_r \neq 0$. A multi-valued function $F: X \to Y$ is said to be *continuous* at x_o if $\{x_d\} \to x_o$ implies that $F(x_o) =$ cofinal limit $\{F(x_d)\} =$ residual limit $\{F(x_d)\}$. The set of all non-null closed subsets of Y is denoted by $S(Y)$. If U and V are open sets in Y define $N(U, V) = \{A \in S(Y) \mid A \subset U \text{ and } A \cap V \neq 0\}$. The set of all $N(U, V)$ such that U and V are open in Y will be used as a subbasis for the open sets in $S(Y)$. This is equivalent to the topology of Frink [1]. A multi-valued function $F: X \to Y$ is said to be *point closed* if x in X implies that $F(x)$ is a closed set.

Lemma 1. (Fundamental lemma on continuity of multi-valued functions.) *Let X be a Hausdorff space, Y a compact Hausdorff space, and F a point-closed multi-valued function from X to Y. Define f from X to $S(Y)$ by $f(x) = F(x)$. Then F is continuous under the definition above if, and only if f is continuous in the usual sense. The single valued function f is referred to as the induced function of F.*

Lemma 2. *A point-closed multi-valued function F from a Hausdorff space X to a compact Hausdorff space Y is continuous if, and only if*

1. *V open in Y implies that $F^{-1}(V)$ is open in X and*
2. *V open containing $F(x_o)$ implies that there exists an open set U containing x_o such that $F(x)$ is contained in V whenever x is an element of U.*

Lemma 3. *If $F: X \to Y$ is continuous and Y is a T_1 space then F is point-closed.*

Lemma 4. *If $F: X \to Y$ and $G: Y \to Z$ are continuous and X, Y, and Z are compact Hausdorff spaces, then GF is continuous.*

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