BOUNDEDNESS AND CONTINUITY OF LINEAR FUNCTIONALS

BY V. L. KLEE, JR.

1. **Introduction.** One of the most fundamental, familiar, and elementary facts of functional analysis is that if E is a normed linear space and C is its unit cell $\{x: \| x \| \leqq 1\}$, then both of the following statements are true: (a) Every linear functional on E which is continuous is bounded on C. (b) Every linear functional on E which is bounded on C is continuous. From the uniform boundedness principle it follows that a set $C \subset E$ has property (a) if, and only if C is bounded. One purpose of the present paper is to illuminate further this basic question, by characterizing those sets C which have property (b). Since a linear functional is bounded on a set if, and only if, it is bounded on the convex hull of the set, attention is confined to convex sets, the result being as follows:

THEOREM A. *If E is a metric linear space and C is a convex subset of E, the following statements are equivalent*:

(i) *There is a closed linear subspace L of finite deficiency in E such that $C \subset L$ and $C - C$ has non-empty interior relative to L.*

(ii) *Every linear functional on E which is bounded on C is continuous.*

(Except in §5, linear spaces are all over the real field R. A *topological linear space* is a linear space E with an associated Hausdorff topology in which $x + y \mid (x, y) \in E \times E$ and $rx \mid (r, x) \in R \times E$ are both continuous. A *metric linear space* is a topological linear space whose topology is metrizable. The *deficiency* of L in E is the dimension of a linear subspace complementary to L in E. The set $\{x - y: x \in C, y \in C\}$ is $C - C$, $\sim$ being used for set-theoretic subtraction.)

Theorem A is proved in §2, and under stronger hypotheses is extended in §3 as follows:

THEOREM B. *If E is a complete metric linear space and C is a closed convex subset of E, then* (i) *and* (ii) *above are equivalent to*

(iii) *Every linear functional f on E for which $fC \neq R$ is continuous.*

(A *complete metric linear space* is a topological linear space whose topology can be generated by a complete metric. It is known that a topological linear space is a metric linear space if, and only if, it satisfies the first axiom of countability, that a metric linear space admits an invariant metric, and that a complete metric linear space is complete in each admissible invariant metric. (See [4; 33] and [6].) Thus the complete metric linear spaces are the same as Banach's spaces of type (F) [2; 35].)

From the aspect of possible applications, Theorem B is the more significant. Among the many results which guarantee continuity of "non-negative" linear

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