LIMITS FOR THE CHARACTERISTIC ROOTS OF A MATRIX. VI: NUMERICAL COMPUTATION OF CHARACTERISTIC ROOTS AND OF THE ERROR IN THE APPROXIMATE SOLUTION OF LINEAR EQUATIONS

By Alfred Brauer and H. T. LaBorde

This paper is a continuation of former papers [1], (see also [2] and [3]). It can be read without the knowledge of the other parts since only a few of their results will be used. The numeration of the theorems and equations will be continued.

In 1874, L. Seidel [12] published a method for obtaining an approximate solution of a system of simultaneous linear equations of high order. Since then a number of other such methods have been obtained, for example by R. v. Mises and H. Pollaczek-Geiringer [10] and by H. Hotelling [8]. For a complete bibliography see the paper of G. E. Forsythe [5].

In most of these methods it is assumed that the determinant of the coefficients is different from zero and proof for convergency of the iterative process set up is obtained under this assumption. If the system has solutions and if the rank r of the matrix of coefficients and a non-vanishing minor of order r are known, then the unknowns which belong to the remaining $n - r$ columns can be arbitrarily chosen, and an inhomogeneous system for the other r unknowns with non-vanishing determinant is to be solved.

If an approximation has been obtained, then for many applications it is necessary to have estimates for the accuracy of this approximation. H. Wittmeyer [13] published a method which gives estimates for the error in the solution of an inhomogeneous system with non-vanishing determinant. Some years later R. Redheffer [11] obtained independently some of Wittmeyer's results. Wittmeyer's main result is the following theorem.

Given the system L of n inhomogeneous linear equations

$$L_\mu = \sum_{\nu=1}^{n} a_{\mu\nu}x_\nu - b_\mu = 0 \qquad (\mu = 1, 2, \cdots, n) \tag{132}$$

whose coefficient matrix A has a non-vanishing determinant. Let $\mathfrak{x} = (x_1, x_2, \cdots, x_n)$ be the exact solution of (132) and $\mathfrak{x}^* = (x_1^*, x_2^*, \cdots, x_n^*)$ be an approximate solution. We set

$$d_\nu = x_\nu^* - x_\nu \qquad (\nu = 1, 2, \cdots, n). \tag{133}$$

Moreover, we set $L(\mathfrak{x}^*) = \mathfrak{g}$ and denote the conjugate transpose of A by A^*. Then the characteristic roots of A^*A are positive, e.g. by a theorem of E. T. Browne [4]. If λ denotes the smallest of these roots, then

$$| d_\nu | \leq | \mathfrak{g} |/\lambda^{\frac{1}{2}} \qquad (\nu = 1, 2, \cdots, n). \tag{134}$$

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