THE RADIAL VARIATION OF ANALYTIC FUNCTIONS

By Walter Rudin

1. We shall be concerned with functions f which are analytic in the interior U of the unit circle; K will denote the closure of U, and we adopt the notation

$$W(f; r, \theta) = \int_0^r | f'(\rho e^{i\theta}) | \, d\rho, \qquad V(f; \theta) = W(f; 1, \theta). \tag{1}$$

Evidently, $V(f; \theta)$ is the total variation of f on the radius of U which terminates at the point $e^{i\theta}$; geometrically speaking, $V(f; \theta)$ is the length (finite or infinite) of the curve which is the image of this radius under f. We call $V(f; \theta)$ the *radial variation* of f.

If $f(z) = \sum a_n z^n$, and if $V(f; \theta)$ is finite, the series $\sum a_n e^{in\theta}$ is said to be summable $| A |$ (i.e., absolutely Abel-summable). Summability $| A |$ of trigonometric series has been studied by Whittaker [7] and Prasad [4]; the function-theoretic approach comes to the foreground in a more recent paper by Zygmund [9].

2. The principal aim of the present paper is the construction of some counter-examples, i.e., functions for which $V(f; \theta) = \infty$ for almost all θ, but in order to gain better perspective, we first list some positive results.

(a) If $\sum | a_n | < \infty$, then $V(f; \theta)$ is bounded. In fact, it is trivial that $V(f; \theta) \leq \sum | a_n |$.

(b) If the image of U under f has finite area, i.e., if

$$\iint_U | f'(z) |^2 \, d\sigma_z < \infty,$$

where $d\sigma$ denotes 2-dimensional Lebesgue measure, we see with the aid of the Schwarz inequality that $\int_0^{2\pi} V(f; \theta)^2 \, d\theta < \infty$.

(c) If $f^*(\theta) = \lim_{r\to 1} f(re^{i\theta})$, and if $f^* \in \text{Lip } \alpha$ $(0 < \alpha \leq 1)$, then $V(f; \theta)$ is bounded. In fact, the hypothesis implies that $f'(re^{i\theta}) = O((1 - r)^{\alpha - 1})$, [2; 627].

(d) If f^* is of bounded variation in a neighborhood of θ, or if f^* satisfies Dini's convergence test for Fourier series at θ, then $V(f; \theta) < \infty$, [4].

(e) A much deeper result: if f has non-tangential boundary values on a set E, then for almost all $\theta \in E$

$$W(f; r, \theta) = o\left(\log^{\frac{1}{2}} \frac{1}{1 - r}\right);$$

this estimate cannot be improved, even for $f \in H_2$, [9; 196].

Received June 30, 1954.