CONGRUENCES FOR GENERALIZED BELL AND STIRLING NUMBERS

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1. **Introduction.** E. T. Bell [3] has generalized the ordinary Stirling numbers. We may define the numbers $S(c, r, s)$ by means of

$$E_0(uE_{s-1}(x)) = \sum_{r=1}^{\infty} \frac{x^r}{r!} \sum_{c=0}^{r} u^c S(c, r, s), \tag{1.1}$$

where $E_k(x)$ is defined recursively by means of

$$E_0(x) = e^x - 1, \qquad E_k(x) = E_0(E_{k-1}(x)) \qquad (k = 1, 2, \cdots). \tag{1.2}$$

The Bell numbers introduced in [2] may be defined by

$$E_s(x) = \sum_{r=1}^{\infty} \frac{x^r}{r!} B(r, s), \tag{1.3}$$

or what is the same thing

$$B(r, s) = \sum_{c=0}^{r} S(c, r, s). \tag{1.4}$$

The notation $S(c, r, s)$ is due to Becker and Riordan [1]; c, r, s denote column, row and stack (or index), respectively.

In their paper Becker and Riordan derive interesting congruences for $S(c, r, s)$ and $B(r, s)$; special cases of these results have been proved by Bell, Marshall Hall [6] and Touchard [8]. In particular it is proved that for p prime, $B(r, s)$, as a function of r, has period (mod p) equal to

$$p^{p^m} - 1 \qquad (p^{m-1} \leq s < p^m), \tag{1.5}$$

while $S(c, r, s)$ has the period

$$q_m^i(q_m - 1) \qquad (p^{m-1} < s \leq p^m,\ q_m^{i-1}p^s < c < q_m^i p^s), \tag{1.6}$$

where $q_m = p^{p^m}$.

In the present paper we derive similar results (mod p^k). In particular we find that the period (mod p^k) of $B(r, s)$ is (a divisor of)

$$p^{k-1}(p^{p^m} - 1) \qquad (p^{m-1} \leq s < p^m); \tag{1.7}$$

for $s = 1$, there is the slightly better period $p^k(p^p - 1)/(p - 1)$ corresponding to the known result $(p^p - 1)/(p - 1)$ in the case $k = 1$.

The method of proof used here is an elaboration of that of [5] and suggests the following generalization. Let $f(x) = \sum_1^\infty a_m x^m/m!$, where the a_m are rational integers, satisfy

$$f'(x) = 1 + \sum_{m=1}^{\infty} c_m f^m(x), \tag{1.8}$$

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