A GENERALIZATION OF PASCAL'S THEOREM.

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In this journal, N. A. Court [3] has given a generalization of Pascal's theorem in three dimensional space. He therefore stated the theorem for the plane in the following form: If the three pairs of sides of a triangle are cut by three transversals, the six points of intersection lie on a conic, if, and only if, the three transversals cut the respective third sides in three collinear points. The generalization is then given as follows: the four triads of points in which the four triads of coterminous edges of a tetrahedron T are cut by four transversal planes, respectively, lie on the same quadric, if and only if, the four secant planes meet the faces of T opposite the respective vertices in four lines which are either coplanar or hyperbolic. The proof as given by Court is short and to the point, it makes use of geometrical reasoning only.

Court's theorem was found long ago by Chasles [2; 400] and given by him without proof in his classical *Aperçu*. A demonstration may be read in Salmon's textbook [6; 141-142]. Some years ago, Kollros [4] published a further generalization of the theorem—or at least of one part of it—to n-dimensional space. In doing so, one has to consider what the extension in R_n may be of three points in a plane being collinear or four lines in space being (coplanar or) hyperbolic. The natural generalization seems to be what is denoted by Kollros as *espaces associés* and called by Italian geometers [1], [5] the Schäfli position of $(n + 1)$ linear subspaces R_{n-2} in R_n. That position is defined by the property that every line that meets n out of the $n + 1$ subspaces also meets the last one. There is a simple way to express analytically that $n + 1$ subspaces R_{n-2} are in Schläfli position: necessary and sufficient for this is that the Plücker coordinates of the subspaces are linearly dependent. (The theorem is an immediate consequence of the fact that the condition for the intersection of a R_{n-2} and a line is bilinear in the Plücker coordinates of both.) Making use of it we not only give a proof of Kollros' theorem which is similar to his, but we are also able to prove the converse theorem, which starts from the Schläfli position as a given fact and has as a conclusion that the $n(n + 1)$ points on the edges of the simplex lie on a quadratic variety. We have added some remarks for the case $n = 3$.

2. We consider in n-dimensional projective space a simplex $A_1A_2 \cdots A_{n+1}$ and a quadric Q, that intersects the edge A_iA_k in two points denoted by B_{ik} and B_{ki} respectively. We "conjugate" B_{ik} to the vertex A_i and B_{ki} to A_k. Assuming that the quadric does not pass through any of the vertices the n points $B_{i1}, B_{i2}, \cdots B_{in+1}$ conjugated to A_i are different one from another; hence they determine a R_{n-1}, denoted by w_i. The $(n - 1)$ dimensional face of the simplex opposite to A_i is called α_i; w_i and α_i have a $(n - 2)$-dimensional

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