PERIODIC SOLUTIONS OF DIFFERENTIAL EQUATIONS CONTAINING A PARAMETER

By D. C. Lewis

Introduction. Concerning a system of differential equations of the type

$$\frac{dx_i}{dt} = X_i(x_1, \cdots, x_n, \mu, t) = X_i(x, \mu, t) \qquad (i = 1, \cdots, n)$$

in which the X's are sufficiently regular real functions of the indicated arguments and are periodic in t with period 2π (say), Poincaré [5; 82-83, 180-81] has proved that the existence of a known periodic solution, $x_i = x_i(t) = x_i(t + 2\pi)$ when $| \mu | = 0$ implies the existence of periodic solutions for all sufficiently small $| \mu | > 0$, at least if the variational equations (appropriate to the given periodic solution) have no nontrivial periodic solutions with period 2π. This condition we shall refer to as Poincaré's hypothesis H. In the analytic case it is furthermore proved that these solutions admit power series expansions in μ. Unfortunately the usefulness of these important classical results of Poincaré is severely limited by lack of information on how to calculate a definite interval for μ for which these periodic solutions exist. Furthermore, in the analytic case, the fact that convergent series for the periodic solutions exist and that the first few terms may even be written down explicitly is, for some purposes (such as numerical calculation), of limited interest in the absence of information as to how rapidly this convergence takes place.

The purpose of this paper is to obtain definite estimates for the size of the interval for μ and to obtain, in the analytic case, upper bounds for the remainders obtained by cutting the series off at the m-th terms.

An estimate of the size of the μ-interval can be done in many ways. In particular, it is possible to review Poincaré's original proof and obtain such an appraisal in terms of bounds for the X's and their first and second partial derivatives and in terms of a lower bound for the absolute value of a certain determinant, the non-vanishing of which is essentially Poincaré's hypothesis H, previously mentioned. Poincaré used the implicit function theorem, and therefore a "quantitative" form of this theorem is necessarily involved in order to specify definitely the size of the neighborhood in which the implicit functions are defined. Such a form for the implicit function theorem suitable for this purpose is given, for instance, as a lemma on page 129 of [1]. The final theorem on periodic solutions resulting from such an analysis is very awkward and the estimates do not appear to be very sharp. Hence we shall omit results of this type.

It turns out, however, that the method of integral equations and successive

Received November 13, 1953. This research was supported in whole or in part by the United States Air Force through the Office of Scientific Research.