# STRUCTURE THEOREMS FOR MODULAR SUBGROUPS

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**Introduction.** In this paper we consider certain congruence subgroups of the modular group $G$, and derive some theorems which point out their arithmetical structure. Application of these theorems is then made to questions concerning the existence and construction of rational bases for functions invariant with respect to the substitutions of these subgroups.

1. **Preliminary remarks and definitions.** $G$ is the full modular group: i.e., the group of $2 \times 2$ matrices of determinant 1 with rational integral elements. We set

$$S = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

It is well-known that $S$ and $T$ are generators of $G$. We observe that

$$S^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}, \quad T^2 = -I.$$

$G_0(n)$ is the subgroup of $G$ defined as follows: The element

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

of $G$ belongs to $G_0(n)$ if and only if $n \mid c$. Notice that $G_0(1) = G$.

The class of functions that will be considered is the class of functions $g(\tau)$ with polar singularities at most in various uniformizing variables throughout the upper $\tau$ half-plane. The word 'function' is to be understood in this sense in what follows.

We will say that $g$ is a *function on the subgroup* $G_1$ of $G$ if $g$ is invariant with respect to the substitutions of $G_1$, regarded as linear fractional transformations. We will also say that $g$ is *maximal on* $G_1$ if $g$ is a function on $G_1$ but is not a function on any subgroup $G_0$ of $G$ properly containing $G_1$.

We will say that the set of functions $\{g_1, g_2, \cdots, g_n\}$ is a *rational basis* for the subgroup $G_1$ of $G$ if

(a) The functions $g_1, g_2, \cdots, g_n$ are functions on $G_1$.

(b) Any function on $G_1$ is a rational function of $g_1, g_2, \cdots, g_n$.

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