CONGRUENCES FOR THE SOLUTIONS OF CERTAIN DIFFERENCE EQUATIONS OF THE SECOND ORDER

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1. **Introduction.** D. H. Lehmer [3] discussed the difference equation

$$u_{n+1} = (an + b)u_n + cu_{n-1}\ , \tag{1.1}$$

where u_0, u_1, a, b, c, are assigned integers. He proved in particular that, if m is an arbitrary positive integer and $(a, m) = 1$, then

$$u_{n+2m} \equiv c^m u_n \pmod{m}. \tag{1.2}$$

In the present paper we first extend this result in the following manner. It is convenient to take (1.1) in the "normal" form

$$u_{n+1} = nu_n + cu_{n-1}\ . \tag{1.3}$$

Since $(a, m) = 1$ and n is allowed to take on all integral values there is no real loss in generality in discussing (1.3); moreover we also assume $(c, m) = 1$. Define

$$\Delta^r u_n = \sum_{s=0}^{r} (-1)^{r-s}\binom{r}{s}c^{m(r-s)}u_{n+2sm}\ . \tag{1.4}$$

Then we show that

$$\Delta^{2r-1}u_n \equiv \Delta^{2r}u_n \equiv 0 \pmod{m^r} \tag{1.5}$$

for $r \geq 1$, m odd and all n; for even m the modulus is replaced by $2^{-r+1}m^r$.

We next consider certain non-homogeneous equations suggested by the ménage polynomial [2]

$$U_n(t) = \sum_{k=0}^{n} \frac{2n}{2n-k}\binom{2n-k}{k}(n-k)!(t-1)^k \tag{1.6}$$

and the closely related polynomial

$$W_n(t) = \sum_{k=0}^{n} \binom{2n-k+1}{k}(n-k)!(t-1)^k \tag{1.7}$$

which satisfies

$$W_n = nW_{n-1} + (t-1)^2W_{n-2} + 2(t-1)^n. \tag{1.8}$$

In view of (1.3) and (1.8) we consider

$$u_{n+1} = nu_n + bcu_{n-1} + b^n + c^n, \tag{1.9}$$

with $u_0 = 0$, $u_1 = 1$. The symbols b, c denote either indeterminates or rational numbers such that b, c, b^{-1}, c^{-1} are integral (mod m); also we assume that $b \equiv c$

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