THE TRANSLATION PATHOLOGY OF WIENER SPACE

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1. **Introduction.** Translations in Wiener space have been studied by Cameron and Martin [4], Maruyama [10], R. E. Graves [2], and others, and it is known that measurability (but not measure) is preserved under translations by sufficiently smooth functions. This set of smooth functions, however, is a set of measure zero in the space, and it is natural to ask whether a larger class of functions—conceivably all or almost all functions in the space—give translations which preserve measurability. One of the purposes of this paper is to answer this question.

The space C with which we deal is the set of continuous functions $x(t)$ defined on the interval I: $0 \leq t \leq 1$ and vanishing at $t = 0$. Wiener's measure [12; 214–224] in the space is first given on a class of intervals, which are sets of $x \in C$ satisfying

$$\alpha_j < x(t_j) \leq \beta_j , \qquad (j = 1, \cdots, n), \tag{1}$$

where n is any positive integer and the t_j and α_j , β_j satisfy $0 < t_1 < t_2 < \cdots < t_n \leq 1$ and $-\infty \leq \alpha_j \leq \beta_j \leq \infty$ for $j = 1, 2, \cdots, n$. The measure of the interval (1) is given by

$$\frac{1}{(\pi^n t_1(t_2 - t_1) \cdots (t_n - t_{n-1}))^{\frac{1}{2}}} \int_{\alpha_n}^{\beta_n} \cdots \int_{\alpha_1}^{\beta_1} \exp\left\{ -\frac{\xi_1^2}{t_1} - \frac{(\xi_2 - \xi_1)^2}{t_2 - t_1} - \cdots - \frac{(\xi_n - \xi_{n-1})^2}{t_n - t_{n-1}} \right\} d\xi_1 \cdots d\xi_n ; \tag{2}$$

and this definition turns out to be consistent and leads to a totally finite measure on the σ-ring generated by the intervals; the measure of C being unity. We denote the completion of this measure by m_w and we call the sets on which it is defined, "Wiener measurable", (or simply "w-measurable").

As formula (2) suggests, this measure arose from probability considerations; actually from the work of Einstein on the distribution of Brownian motion particles [6]. The question naturally arises whether a simpler measure on the space C might be found which would be defined on all the w-measurable sets (and perhaps others) and would give positive measure to all w-measurable sets of positive measure (and perhaps others); i.e., a measure m satisfying $m_w \ll m$ on the w-measurable sets. This question is of course hard to answer because of the indefiniteness of the word "simpler", but at the present time the

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