# SETS OF CONVERGENCE OF ORDINARY DIRICHLET SERIES

George Brauer

Herzog and Piranian [1] proved that if $M$ is a set of type $F_\sigma$ on the unit circle $C$ then there exists a Taylor series which converges on $M$ and diverges on $C - M$. In the following I shall prove the following analogue for ordinary Dirichlet series.

Theorem. *If $\alpha$ is any real number, and $M$ is a set of type $F_\sigma$ on the line $L$: $\sigma = \alpha$, where $s = \sigma + i\tau$, then there exists an ordinary Dirichlet series $\sum_{n=1}^{\infty} a_n n^{-s}$ which converges on $M$ and diverges on $L - M$.*

*Proof.* If the theorem is proved for $\alpha = 0$, then it can easily be deduced for an arbitrary value of $\alpha$. We shall therefore assume that $L$ is the imaginary axis. If the set $M$ is empty, then the series $\sum_{n=1}^{\infty} (1/n)n^{-s}$ has the desired properties. The series $\sum_{n=2}^{\infty} 1/(n \log^2 n)n^{-s}$ constitutes an example of an ordinary Dirichlet series which has $L$ as its line of convergence and which converges at all points of $L$. It remains to deal with the case where $M$ is neither empty nor equal to the entire $\tau$-axis. In this case $M$ may be written as the union of closed sets $F_p$ ($p = 1, 2, \cdots$), where $F_p \subseteq F_{p+1}$ and $F_1$ is non-empty. For each positive integer $q$ let

$$\mu_{qm} = (-q + 2qm/4^q)\pi \qquad (m = 0, 1, \cdots, 4^q).$$

Let the set $\Lambda_q$ consist of those points $i\mu_{qm}$ on $L$ which satisfy one of the following conditions:

$$(1) \quad \begin{aligned} &q\pi/2 < \Delta(i\mu_{qm}, F_q), \quad q\pi/2^2 < \Delta(i\mu_{qm}, F_{q-1}) \le q\pi/2, \\ &q\pi/2^3 < \Delta(i\mu_{qm}, F_{q-2}) \le q\pi/2^2, \quad \cdots, \quad q\pi/2^q < \Delta(i\mu_{qm}, F_1) \le q\pi/2^{q-1}, \end{aligned}$$

where $\Delta(s, F)$ denotes the distance between the point $s$ and the set $F$. Suppose that $k_q$ of the points $i\mu_{qm}$ are in $\Lambda_q$; let them be denoted by $i\lambda_{qj}$ in accordance with the inequality

$$\lambda_{q1} < \lambda_{q2} < \lambda_{q3} < \cdots < \lambda_{qk_q}.$$

Let

$$C_q(\tau) = 4^{-q} \exp[(-in_q/q)\tau] \sum_{j=1}^{k_q} \exp([-i(j-1)/q]\tau) \sum_{r=0}^{4^q-1} \exp[(-ir/q)(\tau - \lambda_{qj})]$$

$$= \sum_{j=0}^{4^q k_q - 1} a_{q,j} \exp([-i(n_q + j)/q]\tau).$$

The series

$$(2) \qquad \sum_{q=1}^{\infty} C_q(\tau) = \sum_{q=1}^{\infty} \sum_{j=0}^{4^q k_q - 1} a_{q,j} \exp([-i(n_q + j)/q]\tau)$$

constitutes a generalized Dirichlet series (i.e., a series $\sum_{n=1}^{\infty} a_n \exp[-\lambda_n s]$, where $\{\lambda_n\}$ is a sequence of real numbers tending monotonically to infinity), at least

Received November 16, 1953; in revised form, April 24, 1954. The author is indebted to Professor George Piranian for many helpful suggestions.