A NOTE ON THE SERIES $\sum a_n f(nz)$

By J. M. Whittaker

1. Series of the form

$$F(z) \equiv \sum_{n=0}^{\infty} a_n f(nz)$$

are familiar in various connections, $f(z)$ being generally a given function. (An unusual example is the series $\sum a_n \{\zeta(ns) - 1\}$, discussed by Estermann [1].) However, this is not always the case. The problem of determining $f(z)$ in terms of $F(z)$ has been investigated by Hille [2], and shown to lead to the infinite system of bilinear equations, familiar in the theory of Dirichlet series,

$$a_1 b_1 = 1, \qquad \sum_{d \mid n} a_d b_{n/d} = 0, \qquad (n > 1),$$

to which he gave the name *algorithm of Möbius*. My object here is to call attention to a peculiarity in the rate of growth of $F(z)$ in relation to those of the functions $f(z)$ and

$$g(z) \equiv \sum_{n=0}^{\infty} a_n z^n$$

out of which it is compounded. It is evident that the increase of $f(z)$ and $g(z)$ sets an upper limit to that of $F(z)$, since

$$|F(z)| \leq \sum_{n=0}^{\infty} |a_n| M(nr), \qquad (|z| \leq r),$$

where $M(r)$ denotes the maximum modulus of $f(z)$. A more interesting problem is to find, if possible, a lower limit for the rate of growth of $F(z)$. One might suspect that to produce a "small" $F(z)$, one or both of $f(z)$ and $g(z)$ would have to be "small". It will be shown that this is not so. In the example given $f(z)$ and $g(z)$ may both be entire functions of order 1, but $F(z)$ is an entire function of order 0.

2. Let a sequence of integers be defined by the relations

$$n_1 = 2, \qquad n_p = n_{p-1}^p \qquad (p = 2, 3, \cdots), \tag{1}$$

and let a sequence $A_1, A_2, \cdots$ be defined by

$$A_1 = 1, \qquad A_p = -N^{-N}(A_1 n_1^N + A_2 n_2^N + \cdots + A_{p-1} n_{p-1}^N), \tag{2}$$

where N has been written in place of n_p for short. It will be shown that

$$N^{-N} \leq |A_p| \leq (p-1)N^{-N(p-1)/p} \qquad (p \geq 2). \tag{3}$$

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