ENTIRE FUNCTIONS AS LIMITS OF POLYNOMIALS

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1. **Introduction.** Let R be a set of points in the complex z-plane. An R-polynomial is a polynomial whose zeros lie in R. We are interested in the class $C(R)$ of all R-functions, that is, the entire functions $f(z) \not\equiv 0$ which may be obtained as the limit of a sequence $\{f_n(z)\}$ of R-polynomials, the convergence being uniform in every bounded domain (UBD convergence).

The study of $C(R)$ is interesting only if R is an unbounded set. Clearly the limit of a UBD convergent sequence of R-functions belongs to $C(R)$. Thus $C(R) \equiv C(\bar{R})$ so that we may assume that R is closed. Let us consider the example where R is the half-line $\operatorname{Re} z \geq 0$, $\operatorname{Im} z = 0$. In this case $\exp(-\lambda z) \in C(R)$ for every real $\lambda > 0$, for $\exp(-\lambda z) = \lim(1 - \lambda z/n)^n$, where the convergence is uniform in every bounded domain. It is easy to show that in this case $C(R)$ is the class of the functions of the form

$$e^{a+bz} z^m \prod_p (1 - z/z_p),$$

where b is real ≤ 0, m a non-negative integer, $z_p > 0$ and $\sum 1/z_p$ converges. This result is essentially due to Laguerre, who also considered the case where R consists of the entire real axis. Pólya investigated the case of an angle less than π, Pólya and Obrechkoff treated the case of a half-plane. Details of these investigations may be found in Obrechkoff's monograph on the subject [3].

In some previous papers ([1], [2]) I set the problem to investigate $C(R)$ for arbitrary unbounded closed sets R and obtained characterizations of $C(R)$ for "practically all" sets R. These characterizations involve certain relevant geometrical properties of R. An essential part is played by the asymptotic directions and the asymptotes of R, R^2, R^3, etc. (R^2 denotes the set of all points z^2 where $z \in R$, etc.). The case where R consists of an angle greater than π is interesting: in this case $C(R)$ consists of all entire functions $\not\equiv 0$ whose zeros lie in R. A set R with this property will be called *regular*. It was shown that a set R is certainly regular if the asymptotic directions of none of the sets $R^j (j = 1, 2, \cdots)$ lie in a (closed) half-plane.

However, various questions remained. Is it possible that $C(R)$ contains an entire function of infinite order if R is non-regular? The answer given in this paper is no. Again, is it possible that $C(R)$ contains entire functions of arbitrarily large finite order if R is non-regular? The answer to this question turns out to be no also. In other words, if R is not regular, then there is a finite least upper bound $\omega(R)$ to the orders ρ of the functions of $C(R)$.

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