STIELTJES INTEGRAL REPRESENTATION OF CONVEX FUNCTIONS

By W. B. Temple

W. Blaschke and G. Pick [2] showed that a function $f(x)$ which is continuous and convex in the interval $0 \leq x \leq 1$ can be represented by means of a Stieltjes integral. In this paper a similar result is obtained in a different manner. We make use of the Bernstein Polynomials and are able to extend this result to functions which are convex of order k. The generalization to functions of two variables will be treated in a second paper.

We shall employ the following notation:

$$\Delta_\delta^0 f(x) = f(x),$$

$$\Delta_\delta^1 f(x) = \Delta_\delta f(x) = f(x + \delta) - f(x),$$

$$\Delta_\delta^n f(x) = \Delta_\delta^{n-1} f(x + \delta) - \Delta_\delta^{n-1} f(x).$$

Definition 1. We say that $f(x)$ is *convex of order* $k \geq 2$ on the interval $[a, b]$ provided $\Delta_\delta^k f(x) \geq 0$ where $a \leq x < x + k\delta \leq b$.

Definition 2. $P_n(x; f)$ denotes the Bernstein polynomial of $f(x)$ if and only if

$$P_n(x; f) = \sum_{i=0}^{n} \binom{n}{i} x^i (1 - x)^{n-i} f\left(\frac{i}{n}\right).$$

S. Bernstein [1] has proved that these polynomials converge uniformly to $f(x)$ in the interval $0 \leq x \leq 1$ if $f(x)$ is continuous over this interval. We note the following readily established facts for future reference:

$$P_n(0; f) = f(0), \qquad P_n(1; f) = f(1),$$

$$P_n'(x; f) = \sum_{i=0}^{n-1} n\binom{n-1}{i} x^i (1 - x)^{n-i-1} \Delta_{1/n} f\left(\frac{i}{n}\right),$$

$$P_n^{(k)}(x; f) = \sum_{i=0}^{n-k} n(n - 1) \cdots (n - k + 1)\binom{n-k}{i} x^i (1 - x)^{n-i-k} \Delta_{1/n}^k f\left(\frac{i}{n}\right).$$

Lemma 1. *If $f(x)$ is continuous and convex of order 2 in $0 \leq x \leq 1$, then $\{P_n(x; f)\}$ is a monotone non-increasing sequence.*

Proof. We form the difference

$$P_n(x; f) - P_{n+1}(x; f) = \sum_{j=0}^{n-1} \frac{1}{j + 1} \binom{n}{j} x^{j+1} (1 - x)^{n-j} \cdot \left[(j + 1) f\left(\frac{j}{n}\right) - (n + 1) f\left(\frac{j + 1}{n + 1}\right) + (n - j) f\left(\frac{j + 1}{n}\right) \right].$$

Received August 17, 1953; in revised form, February 24, 1954.