# SOME CHARACTERIZATIONS OF VALUATION RINGS

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**Introduction.** Let $K$ be a commutative field and let $K^* = K - \{0\}$ denote the multiplicative group of the non-zero elements of $K$. If $\Gamma$ is a simply ordered Abelian group we say, as usual, that a homomorphism $w$ of $K^*$ onto $\Gamma$ defines a *valuation* of $K$ if $w$ satisfies the inequality $w(a + b) \geq \min (w(a), w(b))$ for all $a, b \in K^*$. Denoting $\Gamma$ additively, the set of all elements $a \in K^*$ such that $w(a) \geq 0$ forms together with the zero-element of $K$ a ring which is the valuation ring of the valuation $w$. In general an integral domain $I$ is said to be a *valuation ring* if the quotient field $K$ of $I$ admits a valuation having $I$ as its valuation ring. We speak of *discrete valuations* and *discrete valuation rings* in the case that the value group $\Gamma$ is discrete in its order topology.

There exist various different characterizations of valuation rings. The most simple one (given by Krull in [4; 165]) says that the integral domain $I$ is a valuation ring if and only if $I$ is simply ordered with respect to the divisibility relation in $I$, i.e. for any pair of elements $a, b \in I$ we have either $a|b$ or $b|a$. Another characterization of Krull (see [4; 168]) says that $I$ is a valuation ring if and only if the non-units of $I$ form an ideal and that every ring lying properly between $I$ and its quotient field $K$ contains the inverse of a non-unit in $I$. A simple characterization was recently given by L. Fuchs [3] stating that $I$ is a valuation ring if and only if every ideal in $I$ is irreducible.

We shall in the present paper give various characterizations of general and discrete valuation rings using the concepts of the theory of $r$-ideals as developed by Prüfer [8], Krull [5] and Lorenzen [6], [7]. We shall study the condition: Every $r_1$-ideal in $I$ is an $r_2$-ideal for different special values of $r_1$ and $r_2$ so as to obtain characterizations of the form: $I$ is a general (resp. discrete) valuation ring if and only if every $r_1$-ideal in $I$ is an $r_2$-ideal.

1. **Ordered Abelian groups.** We shall in this section collect some definitions concerning ordered groups which will be needed in the sequel. All groups considered here will be supposed to be Abelian and will be written multiplicatively. The group $G$ is said to be a *quasi-ordered group* if there is defined a binary relation $\leq$ in $G$ such that (1) $a \leq a$, (2) $a \leq b$ and $b \leq c \rightarrow a \leq c$ and (3) $a \leq b \rightarrow ac \leq bc$ for any $c \in G$. If in addition $a \leq b$ and $b \leq a \rightarrow a = b$ $G$ is called a *partially ordered group*. As usual we write $a < b$ when $a \leq b$ but $a \neq b$. A quasi-ordered group $G$ is said to be *directed* if for any two elements $a, b \in G$ there exists an element $c \in G$ such that $a \leq c$ and $b \leq c$. If $G$ is a partially ordered group such that either $a \leq b$ or $b \leq a$ holds for any two elements

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