A CLASS OF FUNCTIONS RELATED TO HARMONIC FUNCTIONS

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A function $u(x, y)$, harmonic in a region $\mathcal{R}$, has a number of important properties. Of these we cite two that concern the present work:

(i) The Gauss mean value theorem. *For every circle C lying with its interior in $\mathcal{R}$, the value of u at the center is the average of its values on C:*

$$u(\xi, \eta) = \frac{1}{2\pi} \int_0^{2\pi} u(\xi + a \cos t, \eta + a \sin t)\, dt. \tag{1}$$

Here C has its center at (ξ, η), and its radius is a.

(It is known conversely that if the function u is continuous in a region $\mathcal{R}$ and satisfies (1) for all C in $\mathcal{R}$ then u is harmonic.)

(ii) Theorem. *u and all its derivatives are analytic functions of the real variables x, y in $\mathcal{R}$, so if $(\xi, \eta) \in \mathcal{R}$, then the following power series are convergent in some neighborhood of (ξ, η):*

$$u_{p,q}(x, y) = \sum_{r,s=0}^{\infty} \frac{1}{r!s!} u_{p+r,q+s}(x - \xi)^r (y - \eta)^s \qquad (p, q = 0, 1, \cdots). \tag{2}$$

Here we have set

$$u_{ij}(x, y) = \frac{\partial^{i+j} u(x, y)}{\partial x^i\, \partial y^j}, \qquad u_{ij} = u_{ij}(\xi, \eta).$$

We propose to weaken condition (i) and to examine the class of functions satisfying (ii) and the revised (i).

Let C: $(x - \xi)^2 + (y - \eta)^2 = a^2$ be *a fixed circle*, given once and for all, and let K be its interior and $\bar{K}$ the closure: $\bar{K} = K \cup C$.

Definition. (a) A function $u(x, y)$ belongs to class $\mathcal{P}(u \in \mathcal{P})$ if u is analytic in K and if all the series (2) $(p, q = 0, 1, \cdots)$ converge uniformly on $\bar{K}$. (b) $u(x, y)$ belongs to class $\mathfrak{Q}(u \in \mathfrak{Q})$ if $u \in \mathcal{P}$ and if the mean value property (1) holds for u and all its derivatives for the (fixed) circle C.

It is clear that if $u \in \mathcal{P}$ and is harmonic in K then $u \in \mathfrak{Q}$. For, each derivative is continuous on $\bar{K}$ and is therefore expressible by means of its Poisson integral taken over C; and choosing $(x, y) = (\xi, \eta)$, this integral reduces to the mean value relation (1). Hence $u \in \mathfrak{Q}$. It will be shown later that $\mathfrak{Q}$ contains non-harmonic functions also. In the meantime we shall examine properties of class $\mathfrak{Q}$.

From an elementary relation for binomial coefficients one readily establishes

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