CLASS NUMBER RELATIONS FOR QUADRATIC FORMS OVER $GF[q, x]$

By G. Cleaves Byers

1. **Introduction.** In his investigations of elliptic functions with complex multiplication Kronecker discovered certain remarkable formulae expressing the sums of class numbers of definite quadratic forms for a certain set of related discriminants in terms of simple divisor functions. He subsequently gave purely arithmetic proofs of these formulae [6] by considering the equivalence of bilinear forms and that of definite quadratic forms of related discriminant under a somewhat restricted definition of equivalence.

Let p be an odd prime, n a positive integer and $q = p^n$; then $GF(q)$ denotes the Galois field of order q and $GF[q, x]$ denotes the ring of polynomials in the single indeterminate x with coefficients from $GF(q)$.

In this paper we develop formulae, analogous to those of Kronecker, for the sums of class numbers of definite quadratic forms with coefficients from $GF[q, x]$. To this end we study the theory of equivalence of bilinear and quadratic forms with polynomial coefficients under appropriate definitions of equivalence and definiteness. Typical of the results obtained is this: if ∇ is a polynomial of $GF[q, x]$ of odd degree $2m + 1$, and $h(\Delta)$ is the number of classes of equivalent quadratic forms of discriminant Δ, the relation

$$\sum_{\deg R \leq m} h(R^2 - \nabla) = 2 \sum_{\substack{K \mid w \\ \deg K > m}}{}' q^k$$

holds, where the sum on the left extends over all polynomials R of degree not greater than m including $R = 0$; the sum on the right extends over all primary divisors K of ∇, for which k, the degree of K, exceeds m. This relation, together with two results of like character, constitute the principal results obtained here.

2. **Notation and preliminary notions.** Except where otherwise specified, lower case Greek letters will be used to denote elements of $GF(q)$, lower case italic letters to denote positive integers. We will use the upper case italic letters $A, B, \cdots, L$ except H to represent elements of $GF[q, x]$; if $C = \gamma_c x^c + \gamma_{c-1}x^{c-1} + \cdots + \gamma_0$, $\gamma_c \neq 0$, we say that c is the *degree* of $C (c = \deg C)$; in general, the lower case italic letter will indicate the degree of the corresponding polynomial. We define $| C | = q^c$. The *signum* of C is $\gamma_c (\operatorname{sgn} C = \gamma_c)$. If $C = 0$, we say that the degree of C is $-\infty$. We note that $| AB | = | A | \cdot | B |$, $| A + B | \leq \max(| A |, | B |)$. If $| A | < | B |$, $\operatorname{sgn}(A + B) = \operatorname{sgn} B$; if $| A | = | B |$, $\operatorname{sgn}(A + B) = \operatorname{sgn} A + \operatorname{sgn} B$.

Received June 17, 1953. This paper is a portion of a doctoral thesis submitted to Duke University.