INDECOMPOSABLE REPRESENTATIONS AT CHARACTERISTIC p

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1. The question whether the degrees of the indecomposable modular representations of a group were bounded was posed to the author by Professor Zassenhaus. In this note we shall prove that at prime characteristic p every indecomposable representation of a finite group is equivalent to a component of a representation induced by an indecomposable representation of a p-Sylow subgroup. A consequence is that every group with cyclic p-Sylow subgroups has only finitely many classes of indecomposable representations. This property characterizes groups with cyclic p-Sylow subgroups. In fact, we shall prove that every group with non-cyclic p-Sylow subgroups has indecomposable representations of arbitrarily high degree at characteristic p.

2. It will be convenient to work with representation modules rather than representations. Let G be a finite group, F a field of characteristic $p > 0$. We shall use the term G-F-module to describe those modules which correspond to representations of G by non-singular matrices with coefficients in F. In particular, a G-F-module is a finite dimensional vector module over F, and the identity element of G operates as the identity endomorphism. We shall call two G-F-modules M, N equivalent, and write $M \simeq N$, if there exists a G-F-isomorphism of M onto N.

Let S be a subgroup of G, m an S-module. The induced G-F-module is the module m^G of all formal sums $\sum x \cdot u_x$, u_x in m, where the summation extends over all x in a set L of left representatives for G over S. Each element g in G may be written uniquely as a product $g = g^+ g_+$, with g^+ in L and g_+ in S. Operators for G and F on m^G are defined respectively by

$$g \sum x \cdot u_x = \sum (gx)^+ \cdot (gx)^+ u_x \qquad (g \text{ in } G)$$

$$(\sum x \cdot u_x) = \sum x \cdot u_x \omega \qquad (\omega \text{ in } F).$$

(Compare [2; §2]). This construction is independent, up to equivalence, of the choice of representatives. For convenience we shall assume that $L \cap S = 1$, so that for s in S, $s^+ = 1$, $s_+ = s$. If m has dimension k over F then m^G has dimension $k[G:S]$ over F, where $G:S$ is the index of S in G.

A G-F-module M of dimension l over F induces an S-F-module M_S also of dimension l over F, obtained by considering only the operators of S on M.

We shall require the following lemma, which is a corollary to [1; 6.3(c)] or [2; Theorem 2(b)]. We indicate a direct proof in order to make the present note self-contained.

Received April 2, 1953. Parts of this note were presented to the American Mathematical Society, April 24, 1953. The author is a National Research Fellow.