# MODULES WITH A GROUP OF OPERATORS

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1. Let $G$ be a group, $\Omega$ a set. For a subgroup $S$ of $G$, let $M_S$ denote the $S$-$\Omega$-submodule induced by a given $G$-$\Omega$-module $M$. Assuming that the index $G{:}S$ is finite, we shall characterize those $G$-$\Omega$-modules $M$ such that $M$ is a direct summand of every $G$-$\Omega$-module $H$ of which it is a $G$-$\Omega$-submodule in such a way that $M_S$ is a direct summand of $H_S$. In case $S = 1$ our result coincides with W. Gaschutz's generalization of Maschke's theorem [3]. Since, as we shall prove, the $G$-$\Omega$-module $M$ with the above property are precisely those such that the $G$-$\Omega$-module induced by $M_S$ contains a direct summand equivalent with $M$, our result contains a theorem of B. Eckmann [1], [2].

As an application we obtain that a representation submodule $M$ of a representation module $H$ for a finite group $G$ in a field of prime characteristic $p$ is a direct summand of $H$ if and only if $M_S$ is a direct summand of $H_S$ with $S$ a $p$-Sylow subgroup of $G$.

2. We shall call a module $M$ a $G$-$\Omega$-module if it admits the group $G$ as a group of left operators and the set $\Omega$ as a set of right operators such that $g(u\omega) = (gu)\omega$, for all $g$ in $G$, $u$ in $M$ and $\omega$ in $\Omega$. If $M$ and $N$ are $G$-$\Omega$-modules, we shall call $M$ equivalent to $N$, and write $M \simeq N$, if there exists a $G$-$\Omega$-isomorphism of $M$ onto $N$.

Let us choose once and for all a set $L$ of left representatives for $G$ over its subgroup $S$. Then each element $g$ in $G$ can be written uniquely as a product $g = g^+g_+$, with $g^+$ in $L$ and $g_+$ in $S$. We assume that the index $G{:}S$ is finite, so that $L$ is a finite set. For simplicity we shall assume that $L \cap S = 1$, then for $s$ in $S$, $s^+ = 1$, $s_+ = s$.

We denote by $m^G$ the $G$-$\Omega$-module induced by a given $S$-$\Omega$-module $m$. The elements of $m^G$ may be realized as the formal sums

$$\sum x\cdot u_x \qquad (u_x \text{ in } M),$$

where the summation is to extend over all $x$ in $L$, with addition defined by

$$\sum x\cdot u_x + \sum x\cdot v_x = \sum x\cdot(u_x + v_x)$$

and with operators for $G$ and $\Omega$ defined respectively by

$$g(\sum x\cdot u_x) = \sum (gx)^+\cdot(gx)_+u_x \qquad (g \text{ in } G)$$

and

$$(\sum x\cdot u_x)\omega = \sum x\cdot u_x\omega \qquad (\omega \text{ in } \Omega).$$

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