AUTOMORPHISMS GENERATED BY A CLASS OF SUBNORMAL SUBGROUPS

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1. **Introduction.** It is well known [2; 78] that every normal subgroup of a group G gives rise to an Abelian group of automorphisms of G. We shall show that this is but a special case of the following: to each of a large class of subnormal subgroups of G is associated an ascending chain of groups of automorphisms, the first of which is nilpotent, the rest solvable; and the number of groups in this chain depends upon how much of the ascending central series of G lies in the subnormal subgroup. Roman capitals shall denote groups, subgroups or subsets of a group; lower case Roman letters, late in the alphabet, shall denote group elements; h, i, j, k, s and t are to be non-negative integers reserved for indexing and lower case Greek letters are to be used for automorphisms. Groups are to be written multiplicatively with unity e, and (e) is to be the subgroup consisting of e alone. If U and V are normal subgroups of G, then $U \div V$, the commutator quotient [1] of U by V, is to be the set of all $x \varepsilon G$ with $[x, v] = x^{-1}v^{-1}xv \varepsilon U$, for every $v \varepsilon V$. $U \div V$ is itself a normal subgroup of G.

For a group G, $Z_1(G)$ is to be the center of G, whence $Z_1(G) = (e) \div G$; and $Z_{k+1}(G) = Z_k(G) \div G$, so that the ascending central series [2; 46] of G: $Z_0(G) = (e) \subset Z_1(G) \subset Z_2(G) \subset \cdots$ is described. G is nilpotent of class $\leq k$ if $G = Z_k(G)$. For normal subgroups U and V of G, $[U, V]$ is to be the subgroup generated by all $[u, v]$, $u \varepsilon U$, $v \varepsilon V$. Define $D(G, 1) = [G, G] \supset D(G, 2) = D(D(G, 1), 1) \supset \cdots \supset D(G, k + 1) = D(D(G, k), 1) \supset \cdots$; and we have described the higher commutator series for G. If $D(G, k) = (e)$, we say that G is $\leq k$-step solvable. It should be noted that the set inclusion signs $\supset$ and $\subset$ do not preclude equality.

2. Automorphisms related to a normal subgroup.

Lemma 1. *Let U be a normal subgroup of a group G. Let α be an automorphism of G such that α induces the identity automorphism on G/U and such that $\alpha(u) \equiv u \bmod Z_j(G)$ for every $u \varepsilon U$. Then $\alpha(x) \equiv x \bmod U \cap (Z_j(G) \div U)$ for every $x \varepsilon G$, and α induces the identity automorphism on $G/Z_{j+1}(U)$.*

Proof. For $x, y \varepsilon G$, $\alpha(xy) = xyu(xy) = xu(x)yu(y)$, where $u(xy), u(x), u(y) \varepsilon U$. Hence $yu(xy) = u(x)yu(y)$. If $x \varepsilon U$, $u(x) \varepsilon Z_j(G)$, and $u(x)y \equiv yu(x) \bmod Z_{j-1}(G)$ if $j \geq 1$, and $u(x)y = y = yu(x)$ if $j = 0$. Hence, for $x \varepsilon U$, $u(xy) \equiv u(x)u(y) \bmod Z_{j-1}(G)$ if $j \geq 1$; and $u(xy) = u(x)u(y)$ if $x \varepsilon U$ and if $j = 0$. If $y \varepsilon U$, $x \varepsilon G$, there exists $y' \varepsilon U$ with $xy = y'x$, since U is normal. Then $u(x)yu(y) = yu(xy) = yu(y'x) \equiv yu(y')u(x) \bmod Z_{j-1}(G)$ if $j \geq 1$; and

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