A THEOREM ON GENERALIZED CONJUGATE NETS IN PROJECTIVE n-SPACE

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A theorem proved in a recent paper by the author [1] is the following:

In a linear space S_n of n (≥ 3) dimensions, let N_x be a conjugate (parametric) net. Let M, M' be points on the u-, v-tangents at x of the net N_x, respectively, which describe two nets N_M, $N_{M'}$ having the property that the tangent plane of N_M ($N_{M'}$) at M (M') passes through M' (M). The nets N_M, $N_{M'}$ are conjugate nets and each one of them is a Laplace transformed net of the other one.

Cartan [2] has studied r-dimensional varieties in S_n which sustian generalized conjugate nets. The purpose of the present paper is to extend the above stated theorem so that the result shall apply to the r-dimensional varieties of Cartan. The theorem stated above is the special case in which r is equal to 2.

Before stating the theorem to be proved, it will be necessary to define the varieties of Cartan and to state a notable geometrical property of these varieties which generalizes a well-known property of conjugate nets of surfaces in S_n.

Let M_0 denote a generic point of an analytic variety V_0 of r dimensions in $S_n(r \leq n - 1)$. Let the vertices of a local reference frame be points denoted by $M_0, M_1, \cdots, M_n$ of which $M_1, M_2, \cdots, M_r$ are located on the parametric $u^1, u^2, \cdots, u^r$ tangents to V_0 at M_0, respectively. The general coordinates of the vertices satisfy a system of partial differential equations of the form

$$\frac{\partial M_i}{\partial u^\alpha} = \Gamma^h_{i\alpha} M_h \qquad (\alpha = 1, 2, \cdots, r) \tag{1}$$

in which (according to a convention to be adopted throughout the paper) the repeated Latin index in a term denotes summation of all such terms for the index values of the range $0, 1, 2, \cdots, n$, and in which by proper choice of proportionality factors for $M_1, M_2, \cdots, M_n$ the coefficients may be made to satisfy the relations

$$\Gamma^j_{0\alpha} = \delta^j_\alpha, \qquad \Gamma^0_{0\alpha} \neq 0 \qquad (\alpha = 1, 2, \cdots, r; j = 1, 2, \cdots, n).$$

A Greek index will usually have the range $1, 2, \cdots, r$ except that in certain specified instances in the paper it will be restricted to a part of this range. A repeated Greek index will denote summation with respect to the indicated range except when a restriction is specified.

The net of quadric cones of vertex M_0 defined by the equation

$$(\lambda_{r+1}\Gamma^{r+1}_{\alpha\beta} + \lambda_{r+2}\Gamma^{r+2}_{\alpha\beta} + \cdots + \lambda_n\Gamma^n_{\alpha\beta})\, du^\alpha\, du^\beta = 0 \tag{2}$$

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