RAISING IDEMPOTENTS

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1. **Summary.** The classical task of raising orthogonal idempotents from a ring modulo its radical into the ring itself is possible under fairly broad hypotheses (for example, nil radical) for a denumerable number of idempotents. In §3 we show that hypotheses of this type are not sufficient for raising a non-denumerable set of orthogonal idempotents. As a corollary we conclude that Wedderburn's Principal Theorem (every finite-dimensional algebra which is separable modulo its radical is a vector space direct sum of its radical and a semisimple subalgebra) cannot be extended to algebraic or even locally finite algebras without further hypotheses (compare also [3], [4], and our comments in §3). §4 contains several kinds of sufficient conditions for raising orthogonal idempotents, all involving some kind of linear compactness, at least implicitly; one interesting sufficient condition is finite dimensionality of the radical. §5 applies these results to extend Wedderburn's theorem to certain infinite dimensional algebras and, in particular, to compact rings of prime characteristic.

2. **Introduction.** A. Let R be a ring, N an ideal in R, $\{u_i \mid i \varepsilon \mathfrak{J}\}$ a set of orthogonal idempotents in $R - N$ (that is, $u_i^2 = u_i$ and $u_i u_j = 0$ when $i \neq j$). We do not assume that the set $\mathfrak{J}$ of indices is denumerable. We shall say that $\{u_i\}$ *can be raised into* R in case there exists a set $\{e_i\}$ of orthogonal idempotents in R with $e_i + N = u_i$ for each i in $\mathfrak{J}$.

B. The major condition we shall have occasion to use in our sufficiency theorems is linear compactness or compactness relative to a collection Ω of operators. Specifically, suppose R is a topological Abelian group (R may be discrete, in which case all topology vanishes from consideration) and Ω is a collection of endomorphisms (not necessarily continuous) of R. We call R *Ω-compact* provided: whenever a collection of cosets of closed Ω-subgroups has the finite intersection property then the collection has a nonvoid intersection.

C. Let R be Ω-compact. Suppose we are given a collection of equations $\{T_i(x) = a_i\}$ with each a_i in R, each T_i a continuous Ω-endomorphism of R, and suppose every finite subset of these equations can be solved for x in R. Then there is an x in R satisfying all of the equations simultaneously. For if x_i is a solution of $T_i(x) = a_i$ and K_i is the kernel of T_i, then we are asking for an x in $\bigcap_{i \varepsilon \mathfrak{J}} (x_i + K_i)$, where $\{x_i + K_i\}$ is a collection of closed Ω-subgroups with the finite intersection property.

D. If R is Ψ-compact and $\Psi \subset \Omega$ then R is Ω-compact.

E. If R is compact and Ω is arbitrary, then R is Ω-compact.

F. We shall call a topological ring R *linearly compact* in case it is Ω-compact

Received April 22, 1953.