A COHOMOLOGICAL DEFINITION OF DIMENSION FOR LOCALLY COMPACT HAUSDORFF SPACES

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1. Introduction. In 1947 Alexandroff [2] proved a theorem (here rephrased and not stated in its full generality) which says that if X is a compact Hausdorff space, then the Lebesgue covering dimension of X (cov X) is $\leq n$ if and only if for each closed set C in X and e in $\check{H}^n(C)$ there is an extension of e to $\check{H}^n(X)$ (where $\check{H}^n(X)$ is the n-th Čech cohomology group with the integers as coefficients). This result (which incidentally is a generalization of a similar theorem for compact metric spaces also due to Alexandroff [1]), suitably modified, is the basis of the definition of dimension used here. Our dimension function, distinguished by the term "codimension" (abbreviated "cd"), is shown to be well-defined for all locally compact Hausdorff spaces and to satisfy the monotone property, the sum theorem, a product theorem, and several other properties desirable for a dimension function.

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2. Preliminaries. Let G be a fixed non-zero additive Abelian group. Let X be a compact Hausdorff space and X_0 a closed subset thereof. For any subset, A, of X we denote the n-th Alexander-Kolmogoroff cohomology group of A relative to $A \cap X_0$ with G as coefficient group by $H^n(A, A \cap X_0)$ or more concisely by $H_0^n(A)$. Also if $e \in H_0^n(A)$ and $B \subset A$, then $e \mid B$ and $e \mid_0 B$ stand for the natural homomorphic images of e in $H^n(B)$ and $H_0^n(B)$ respectively.

2.1 Definition. The pair (X, X_0) is an element of $D^n(G)$ if and only if for each closed set C in X and e in $H_0^n(C)$ there is an extension of e to $H_0^n(X)$.

Note that if $X_0 = \square$ (the null set) and G is the group of integers, we have by Alexandroff's theorem (since Spanier [10] has shown that the Čech and Alexander-Kolmogoroff groups are isomorphic) that $(X, X_0) \in D^n(G)$ if and only if $\text{cov}(X) \leq n$.

2.2. Lemma. *If X' is a closed subset of X and (X, X_0) is in $D^n(G)$, then $(X', X' \cap X_0)$ is in $D^n(G)$.*

Proof. X' being closed is compact, and if A is any closed set in X', it is also closed in X. Let e be any element of $H_0^n(A)$. Since $(X, X_0) \in D^n(G)$, there is an element e_1 in $H_0^n(X)$ such that $e_1|_0 A = e$. Let $e_2 = e_1|_0 X'$; then we have the extension, for $e_2|_0 A = (e_1|_0 X')|_0 A = e_1|_0 A = e$.

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