GENERALIZED LAPLACIANS OF THE SECOND KIND AND DOUBLE TRIGONOMETRIC SERIES

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1. **Introduction.** Previously, the author [2] has used generalized Laplacians of the first kind to give a necessary and sufficient condition that there exists a Riesz mean which sums a given double trigonometric series in a circular manner. Using generalized Laplacians of the second kind, the necessary condition for Riesz summability can be considerably strengthened. It is the aim of this paper to develop this necessary condition. The result to be presented here will be analogous to that obtained by Plessner in single trigonometric series (see Zygmund [3; 260]).

2. **Generalized r-th Laplacians.** Given $f(x, y)$ integrable in the neighborhood of a point (x_0, y_0), we shall say that f has a generalized r-th Laplacian of the second kind at (x_0, y_0) equal to α_r if

$$\frac{1}{\pi t^2} \int_0^t \rho \, d\rho \int_0^{2\pi} f(x_0 + \rho \cos \theta, y_0 + \rho \sin \theta) \, d\theta$$

$$= \alpha_0 + \frac{\alpha_1 t^2}{2[2!]^2} + \cdots + \frac{\alpha_r t^{2r}}{(r+1)[2^r r!]^2} + o(t^{2r}),$$

where $t > 0$ and α_i $(i = 0, \cdots, r)$ are constants. If, furthermore, $f(x, y)$ is integrable on the circumference of every circle contained in this neighborhood with (x_0, y_0) as center, we shall say that f has a generalized r-th Laplacian of the first kind equal to β_r if

$$\frac{1}{2\pi} \int_0^{2\pi} f(x_0 + t \cos \theta, y_0 + t \sin \theta) \, d\theta = \beta_0 + \frac{\beta_1 t^2}{[2!]^2} + \cdots + \frac{\beta_r t^{2r}}{[2^r r!]^2} + o(t^{2r})$$

where $t > 0$ and β_i are constants.

We shall designate the generalized r-th Laplacians of the first and second kind at the point (x_0, y_0) by $\Delta_{1,r} f(x_0, y_0)$ and $\Delta_{2,r} f(x_0, y_0)$ respectively. From the definition it is clear that the existence of $\Delta_{2,r} f(x_0, y_0)$ implies the existence of $\Delta_{2,s} f(x_0, y_0)$ for $0 \leq s \leq r$, the same fact being also true for $\Delta_{1,r} f(x_0, y_0)$ and $\Delta_{1,s} f(x_0, y_0)$. It is also clear that if $f(x, y)$ is integrable in the neighborhood of (x_0, y_0), the existence of $\Delta_{1,r}$ implies the existence of $\Delta_{2,r}$.

Designating the ordinary r-th Laplacian of f if it exists at the point (x_0, y_0) by $\Delta^r f(x_0, y_0)$ where $\Delta^r f = \Delta(\Delta^{r-1} f)$, $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$ and Δ^0 is the identity operator, we have from a simple application of Taylor's expansion in two variables the following theorem:

Received March 19, 1953.