A RECENT NOTE OF KOLBINA

By James A. Jenkins

1. In a recent note Kolbina [6] has indicated the solution of several extremal problems involving univalent functions. His principal tool is Golusin's form of the variational method. We will show here how these same problems can be solved by straightforward considerations using quadratic differentials. It should be noted that many years ago Grötzsch [1] treated very closely related problems. The qualitative aspects of the present problems are almost self-evident and the author has been aware of them for some years. Much of the interest in Kolbina's results lies in the fact that it is possible to obtain such explicit solutions to these problems.

The first problem treated is as follows. Let a_1, a_2 be distinct finite complex numbers. Let $f_1(z)$, $f_2(z)$ be functions regular and univalent for $|z| < 1$, with $f_i(0) = a_i$ $(i = 1, 2)$ and such that the images of $|z| < 1$ by the two functions, taken in a common plane, do not overlap. It is then desired to find the maximum of $|f_1'(0)|^{\alpha} |f_2'(0)|^{\beta}$ $(\alpha > 0, \beta > 0)$ as $f_1(z)$, $f_2(z)$ run over the family of all pairs of functions as above.

We first observe that there is no loss of generality to assume $a_1 = 1$, $a_2 = -1$. Indeed any other situation can be reduced to this by Euclidean transformations. The maximum in the general case is obtained from that in the special case by multiplying by $|(a_1 - a_2)/2|^{\alpha+\beta}$.

2. Let us now set up in this case a family of pairs of functions which will provide the extremal functions in the above problem. In the w-plane regard the quadratic differential $-(w + w^*)dw^2/(w - 1)^2(w + 1)^2$, where w^* is real and $|w^*| > 1$. This quadratic differential has a simple zero at $w = -w^*$, a simple pole at $w = \infty$ and double poles at $w = +1, -1$. Let us denote it for short by $Q(w)dw^2$. The curves on which $Q(w)dw^2 > 0$ will be, as usual, called trajectories. The curves on which $Q(w)dw^2 < 0$ will be called orthogonal trajectories. The differentials corresponding to a pair of values of w^* equal apart from sign give rise to configurations differing only by reflection in the imaginary axis. Thus it will be enough to confine our attention to the case $w^* > 1$. In this case the simple zero lies on the negative real axis. One trajectory ray T_1 emerging from it runs out along the negative real axis to ∞. The other two meet to form a closed curve T_2 which encloses $w = -1$ but not $w = +1$.

Together T_1 and T_2 divide the w-plane into two simply-connected domains B_1 and B_2 containing respectively $w = +1$ and -1. All trajectories in B_1 are Jordan curves enclosing $w = +1$; all trajectories in B_2 are Jordan curves enclosing $w = -1$. We note that, under our assumption that $w^* > 1$, B_2 is a