ANOTHER PROOF OF THE PRIME NUMBER THEOREM

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This paper contains another derivation of the prime number theorem from Selberg's asymptotic formula (formula 5 below). For the following, p represents prime numbers, n positive integers, x, y, z positive real numbers, and $R \equiv R_x$ the interval $(\log x, x/\log x)$. As usual, $\vartheta(x) = \sum_{p \le x} \log p$.

Used are the following well known and elementarily provable facts:

(1)
$$\sum_{y<n\le z} 1/n = \log (z/y) + O(1/y).$$

(2) The prime number theorem is equivalent to $\lim_{x\to\infty} \vartheta(x)/x = 1$.

(3)
$$\vartheta(x) = O(x).$$

(4)
$$\sum_{p\le x} \log p/p = \log x + O(1),$$

and therefore

(4′)
$$\sum_{p \varepsilon R_x} \log p/p = \log x + o(\log x),$$

because by (4) the contributions of the two omitted intervals are $O(\log \log x)$.

(5)
$$\vartheta(x) \log x + \sum_{p\le x} \vartheta(x/p) \log p = 2x \log x + o(x \log x),$$

[1; 305-306], and therefore

(5′)
$$\vartheta(x) \log x + \sum_{p \varepsilon R_x} \vartheta(x/p) \log p = 2x \log x + o(x \log x),$$

because by (3) and (4) the two omitted intervals contribute $O(x \log \log x)$.

(6) If $\overline{\lim}\, \vartheta(x)/x = A$, and $\underline{\lim}\, \vartheta(x)/x = a$, then $A + a = 2$ [1].

(7)
$$\vartheta(z) - \vartheta(y) \le 2(z - y) + o(z) \qquad (y < z \le 2y).$$

The last relation can be derived from (5) as follows: write (5) for z and for y; add in the second equation $\vartheta(y) \log (z/y)$ to the left member, and $2y \log (z/y)$ to the right member. Both added terms are $O(y)$ because of (3), and because $z/y \le 2$. Thus

$$\vartheta(z) \log z + \sum_1 = 2z \log z + o(z \log z)$$

$$\vartheta(y) \log z + \sum_2 = 2y \log z + o(z \log z).$$

Now subtract, remember that $\sum_2 \le \sum_1$, and divide by $\log z$.

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