RINGS OF ARITHMETIC FUNCTIONS. II: THE NUMBER OF SOLUTIONS OF QUADRATIC CONGRUENCES

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1. **Introduction.** Let r be a positive integer and F a field of characteristic zero containing the r-th roots of unity. As in [1] we say that a single-valued function $f(n)$ is (r, F) arithmetic, or simply arithmetic, if for every rational integer a, $f(a) \in F$ and $f(a) = f(a')$ if $a \equiv a' \pmod{r}$. We defined the Cauchy product h of two functions f, g by the relation

$$h(n) = f \cdot g = \sum_{n \equiv a+b \pmod{r}} f(a)g(b) \qquad (0 \leq n < r), \tag{1.1}$$

where a and b range over a residue system (mod r) such that $n \equiv a + b \pmod{r}$. It was shown that the set of all (r, F) arithmetic functions, under the operations of function addition and Cauchy multiplication, forms a commutative semisimple algebra $\mathfrak{A}_r(F) = \mathfrak{A}$ which is the direct sum of r fields each isomorphic with F [1; Theorem 2].

Particular attention in [1] was paid to the function

$$c(n, r) = \sum_{\substack{(x,r)=1 \\ 0<x<r}} \epsilon(xn, r), \tag{1.2}$$

where the summation is over a reduced residue system (mod r) and where

$$\epsilon(xn, r) = e^{2\pi i xn/r}. \tag{1.3}$$

The function (1.2) is the familiar Ramanujan sum which is factorable as a function of r,

$$c(n, r_1 r_2) = c(n, r_1)c(n, r_2) \qquad \text{if} \qquad (r_1, r_2) = 1. \tag{1.4}$$

It was shown [1; (3.10)] that the function c is orthogonal under Cauchy multiplication. More precisely, if $d|r$, $e|r$,

$$c_d \cdot c_e = \sum_{n \equiv a+b \pmod{r}} c(a, d)c(b, e) = \begin{cases} rc(n, d) & (d = e) \\ 0 & (d \neq e). \end{cases} \tag{1.5}$$

This result can be interpreted algebraically [1; Theorem 3] to show that the set of all elements $\sum a_d c(n, d)$, where d ranges over the divisors of r, and a_d ranges over F, forms a semisimple subalgebra $\mathfrak{C} \subset \mathfrak{A}$, with orthogonal basis given by the elements $[1/r]c(n, d)$.

The Ramanujan sum is but one of a large class of exponential functions

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