NOTE ON THE SHAPE OF LEVEL CURVES OF GREEN'S FUNCTION

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Certain geometric properties like convexity and starshapedness, if possessed by a region R, are also possessed by the subregions R_r of R bounded by the level curves of Green's function for R. In connection with the study of starshapedness, L. R. Ford has pointed out [2] that if for a specific t_0, $0 < t_0 < 1$, the point $w' = t_0 w$ belongs to R (assumed simply connected) whenever the point w belongs to R, then every region R_r has this same property, the pole of Green's function being taken in $w = 0$. The primary object of the present note is to investigate the corresponding question when linear transformations other than $w' = t_0 w$ are involved. We study (§1) this question for given transformations, (§2) the corresponding question where the subregions of R are bounded by images of oricycles in the map of the unit circle onto R (these images are the limits of level curves of Green's function with variable pole), and (§3) the appropriateness of the precise linear transformations used in §§1 and 2.

1. **Subregions bounded by level loci of Green's function.** If the region R of the extended w-plane contains the point 0: $w = 0$, Green's function $G(w)$ for R with pole in 0 is defined as the unique function harmonic in R and continuous in the corresponding closed region except at 0, zero on the boundary of R, and of the form $u_1(w) - \log | w |$ in a neighborhood of 0, where $u_1(w)$ is harmonic in that neighborhood. Thus $G(w)$ is positive throughout R, except that $G(0)$ is not defined. Each locus $G(w) = -\log r$ $(0 < r < 1)$ consists of one or more analytic curves in R with at most a finite number of multiple points, and the subregion R_r of R defined by the inequality $G(w) > -\log r$ is bounded by this locus, and varies monotonically with r. If R is simply connected, the locus $G(z) = -\log r$ is the image (Kreisbild) of the circle $| z | = r$ when $| z | < 1$ is mapped onto R with the origins corresponding to each other.

If $W = T(w)$ is a linear transformation of w, we say that a region R of the extended w-plane has *property* T provided the point $W = T(w)$ lies in R whenever w lies in R. We prove

Theorem 1. *Let the region R of the extended w-plane have property T, where T is of the form*

$$(1) \qquad T: \frac{W}{W-\alpha} = A \frac{w}{w-\alpha} \qquad (\alpha \neq 0,\ A \neq 1,\ 0 < | A | \leq 1);$$

the case $\alpha = \infty$ is to be interpreted as $W = Aw$. If 0 lies in R and Green's function $G(w)$ for R with pole in 0 exists, then every subregion R_r has property T.

Received January 23, 1953.