# A PROBLEM OF HURWITZ AND NEWMAN

By Jean Dieudonné

1. Hurwitz [3] has determined the maximal sets of matrices $A_k$ of order $n$, having complex elements, and satisfying the conditions

$$A_k^2 = I \tag{1}$$

$$A_h A_k = -A_k A_h \qquad \text{for} \qquad h \neq k. \tag{2}$$

Newman [4] solved the same problem when the equations (1) are replaced by

$$A_k^2 = -I. \tag{3}$$

As an application of the methods I have recently developed for the study of systems of involutive collineations or correlations which are "projectively permutable" ([1] and [2]), I shall treat in this paper the following general problem, of which Hurwitz's and Newman's are very particular cases:

*$K$ being an arbitrary sfield of characteristic $\neq 2$, $E$ an $n$-dimensional right vector space over $K$, $\sigma$ an automorphism of $K$ and $\gamma$ an element of $K$ such that $\gamma^\sigma = \gamma$ and $\xi^{\sigma^2} = \gamma^{-1}\xi\gamma$ for $\xi \in K$, determine the maximal number of semi-linear transformations $u_k$ $(1 \leq k \leq p)$ of $E$, relative to the automorphism $\sigma$, satisfying the relations*

$$u_k^2(x) = x\gamma \qquad \textit{for} \qquad x \in E \qquad \textit{and every } k, \tag{4}$$

$$u_h u_k = -u_k u_h \qquad \textit{for} \qquad h \neq k. \tag{5}$$

2. Let $v_k = u_1^{-1}u_k$ for $k \geq 2$; then the $v_k$ are *linear* mappings of $E$ onto itself, satisfying the following relations

$$v_k^2(x) = -x \qquad (k \geq 2) \tag{6}$$

$$v_h v_k = -v_k v_h \qquad (h \geq 2,\ k \geq 2,\ h \neq k) \tag{7}$$

$$u_1 v_k = -v_k u_1 \qquad (k \geq 2). \tag{8}$$

We now distinguish two cases:

(A) $-1$ *is not the square of any element of* $K$. Let $K_1$ be the quadratic extension of $K$, obtained by adjoining to $K$ an element $i$ such that $i^2 = -1$ (see [2; §4]; $K_1$ is here the tensor product of $K$ and the commutative field $Z(i)$, over the center $Z$ of $K$, and $Z(i)$ is the center of $K_1$). Let $\tau$ be the only automorphism of $K_1$ distinct from the identity and leaving invariant the elements of $K$, that is, such that $i^\tau = -i$; on the other hand, the automorphism $\sigma$ of $K$ can be extended to an automorphism of $K_1$, which we shall again note $\sigma$, by the convention $i^\sigma = i^\tau = -i$; $\sigma$ and $\tau$ obviously commute. We can then consider $E$ as a vector

Received October 27, 1952.