THE ASYMPTOTIC BEHAVIOUR OF POWERS OF MATRICES

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I. The Main Results

1. **Introduction.** Let $A = (a_{\nu\mu})(\nu_1\mu = 1, \cdots, n)$ be an $n \times n$ matrix with real or complex numbers as elements. By $A^* = \bar{A}'$ we denote the conjugate-transpose of A and by tr A the trace $\sum_{\nu=1}^{n} a_{\nu\nu}$ of A. The *norm* (or *absolute value*) $N(A)$ of A is defined by

$$(1) \qquad N(A) = (\text{tr } AA^*)^{\frac{1}{2}} = \left(\sum_{\nu_1\mu=1}^{n} | a_{\nu\mu} |^2\right)^{\frac{1}{2}}.$$

It is well known (J. H. M. Wedderburn [11] or [12; 125]; see also Hardy-Littlewood-Pólya [5; 36]) that

$$(2) \quad N(A + B) \leq N(A) + N(B), \quad N(AB) \leq N(A)N(B), \quad N(\lambda A) = | \lambda | N(A)$$

λ being a scalar and A, B two $n \times n$ matrices.

In (2) of §1 we shall give bounds for the norms of powers $A^p(p = 1, 2, \cdots)$. A *lower* bound can readily be found: by a well-known theorem of I. Schur [7] there exists a unitary matrix U which transforms A to a triangular matrix D. The principal diagonal of D consists of the eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$ of A, not necessarily all distinct, arranged in any desired order. From

$$(3) \qquad U^*AU = D, \qquad U^*U = UU^* = I$$

it follows that

$$U^*A^pU = D^p, \qquad U^*(A^p)^*U = (D^p)^*$$

and

$$\text{tr } A^p(A^p)^* = \text{tr } U^*A^p(A^p)^*U = \text{tr } D^p(D^p)^*.$$

Hence by (1) we have

$$(4) \qquad N(A^p) = N(D^p) \geq \left(\sum_{\nu=1}^{n} | \lambda_\nu |^{2p}\right)^{\frac{1}{2}} \qquad (p = 1, 2, \cdots)$$

with equality for all $p = 1, 2, \cdots$, if and only if A is normal, $A^*A = AA^*$.

Suppose now that A can be transformed to the *diagonal form*

$$X^{-1}AX = \Lambda = \text{diag } (\lambda_1, \lambda_2, \cdots, \lambda_n)$$

Received February 9, 1952; in revised form, October 18, 1952. This paper is part of the thesis for the doctor of philosophy at the University of Basle, Switzerland.