ASYMPTOTIC EXPRESSIONS FOR SOME INTEGRALS WHICH INCLUDE CERTAIN LEBESGUE AND FEJÉR CONSTANTS

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1. **Introduction.** In this article, asymptotic expressions are derived for a number of integrals containing a parameter, ξ or x, as the parameter becomes infinite. The following integrals are typical of those studied here:

$$L_1(\xi,b;g;\tau) = \frac{1}{b}\int_0^{1/\tau(\xi)} \frac{g(\xi t)}{t}\,dt,\ 0 < \tau(\xi) = o(\xi), \qquad (\xi \to \infty), \tag{1.1}$$

$$L(x,b;g) = \frac{1}{b}\int_0^{b} \frac{g[(2x+1)t]}{\sin t}\,dt,\ 0 < b < \pi, \tag{1.2}$$

$$L_E(x,b;g) = \frac{1}{b}\int_0^{b} |\cos t|^x \frac{g[(x+1)t]}{\sin t}\,dt,\ 0 < b < \pi, \tag{1.3}$$

and, letting $\exp(a) = e^a$,

$$L_B^0(x,b;g) = \frac{1}{b}\int_0^{b} \frac{\exp(-2xt^2)g[(2x+1)t]}{t}\,dt,\ 0 < b. \tag{1.4}$$

The function $g(t)$ is assumed to be Lebesgue integrable over $(0,\pi)$, of period π, and such that

$$\int_0^1 \frac{g(t)}{t}\,dt \tag{1.5}$$

exists in some unspecified sense. Occasionally the main results are supplemented by providing somewhat more precise information concerning the error term in the special case in which $g(t)$ is required, in addition, to be even. If $g(t) = |\sin t|$, then $L(n,\frac{1}{2}\pi;g)$ and $L_E(n,\frac{1}{2}\pi;g)$, n an integer, are the Lebesgue constants arising in the theory of Fourier series from summation by ordinary convergence [9; 86] and Euler's $(E,1)$-means, respectively. $L_B^0(x,\frac{1}{2}\pi;g)$, in this case, differs from the Lebesgue constants for summation by Borel's exponential means by an additive error of $O(1/x^{\frac{1}{2}})$.

For $g(t) = \sin^2 t$, $L(n,\frac{1}{2}\pi;g)$ are constants introduced by Fejér [3], who showed that their unboundedness, as n becomes infinite, like that of the Lebesgue constants, also implies the existence of a continuous function whose Fourier series diverges at a point. Gronwall [5; 261] gave a particularly simple treatment

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