THE UNITARY STRUCTURE IN FINITE RINGS OF OPERATORS

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1. **Introduction.** From an algebraic point of view, the study of W^*-algebras of types II and III is complicated at the onset by the absence of minimal projections; algebras of type I, which are rich in analogues of minimal projections (or primitive idempotents), have been analyzed rather completely by techniques going back to Wedderburn. From a measure-theoretic point of view, however, the situation in type II's is not so strange. As it develops, there are intimate connections between the theory of W^*-algebras having no minimal projections (or "non-atomic W^*-algebras") and the theory of non-atomic measure spaces. A pertinent example concerns the (frequently generalized) theorem of Liapounoff [8], which asserts that the range of certain non-atomic vector-valued measures is closed and convex. Translated into a statement about non-atomic W^*-algebras, this theorem shows that the weak closure of the set of projections fills out the positive part of the unit sphere; in turn, the weak closure of the unitary group fills out the entire unit sphere.

We show in this paper how these connections with measure theory can be exploited to characterize the algebraic and spatial types of a finite non-atomic W^*-algebra M in terms of the unitary structure of M. Here, by definition, the unitary structure in M is the group M_U of unitary operators in M viewed as a uniform space in the uniform structure induced by the weak topology. Natural notions of isomorphism and strong isomorphism are defined between unitary structures, and it is proved that every isomorphism (or strong isomorphism) between unitary structures in two finite non-atomic W^*-algebras M and N is induced by a $*$-isomorphism (or unitary equivalence) of the algebras M and N themselves. Oddly enough, this result fails in the atomic case.

2. **Topologies on the unitary group.** We begin with an analysis of certain purely topological properties of the unitary group in a finite W^*-algebra. In particular, among the various reasonable topological restrictions on a group isomorphism between unitary groups in two non-atomic W^*-algebras, we single out one which allows us to extend the isomorphism to a weak homeomorphism between unit spheres. The possibility of such an extension will serve as the basis of subsequent developments.

Notation 2.1. By a *W^*-algebra* M on $\mathfrak{H}$ we mean a weakly-closed self-adjoint algebra of operators (containing the identity I) on a complex Hilbert space $\mathfrak{H}$. M_P and M_U, respectively, will denote the collections of projections and unitary operators in M. A W^*-algebra M is called *finite* if the relations $A^*A = I$ ($A \in M$) and $AA^* = I$ are equivalent, and *countably decomposable* (or

Received April 19, 1952; in revised form, August 12, 1952.