LINEAR DIFFERENTIAL EQUATIONS WITH SMALL PERTURBATIONS

By Choy-tak Taam

1. Introduction. Consider the differential equation

$$(p(x)y')' + q(x)y = f(x), \tag{1}$$

where $p(x)$, $q(x)$ and $f(x)$ are complex-valued functions defined for all $x \geq 1$ and $[p(x)]^{-1}$, $q(x)$ and $f(x)$ belong to $L(1, R)$ for every large R. A solution of (1) is a function $y(x)$ which is absolutely continuous and satisfying the equation (1) for almost all x on $1 \leq x < \infty$ if $p(x)y'(x)$ is replaced by an absolutely continuous function which is equal to $p(x)y'(x)$ almost everywhere on $1 \leq x < \infty$.

The first part of this paper is devoted to the study of certain properties of the solutions of (1). All the conditions assumed are expressed in terms of integrals involving the coefficients of (1). Sufficient conditions for the solutions to satisfy certain order relation will first be given. From this we obtain other properties of the solutions. Theorems 2 and 3 are extensions of the results of E. Hille [4].

The main tool we use in this paper is a lemma due to R. Bellman who uses it to study the boundedness and stability of the solutions of linear differential equations [1], [2], [3].

The results we obtain for equation (1) can be extended to the linear differential equations of the n-th order. However, due to the difficulty of manipulation, only the results for the linear differential equations of the third order shall be given.

With the help of the Phragmén and Lindelöf theorems, our results also can be easily extended to the complex domain.

2. A lemma. We shall need the following lemma to establish our results. It was first proved by R. Bellman. Since the proof is simple, a proof is given here for the sake of completeness.

Lemma. *Let $u(x)$ and $v(x)$ be real-valued functions defined for $x \geq 1$, both being non-negative. Furthermore let $v(x)$ and $u(x)v(x)$ be $L(1, R)$ for every large R and M be a positive constant. If*

$$u(x) \leq M + \int_1^x u(t)v(t)\, dt \qquad (x \geq 1), \tag{2}$$

then

$$u(x) \leq Me^{\int_1^x v(t)dt} \qquad (x \geq 1). \tag{3}$$

Received June 27, 1952.