CONVEX FUNCTIONS AND UPPER SEMI-CONTINUOUS COLLECTIONS

BY R. D. ANDERSON AND V. L. KLEE, JR.

Introduction. A real valued function f is called *convex* if its domain D_f is an open convex subset of Euclidean n-space E^n and $f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y)$ whenever $0 \leq t \leq 1$ and $\{x, y\} \subset D_f$. A *C-collection* is an upper semi-continuous collection of compact convex sets in E^n. We associate with each positive convex f a C-collection P_f, called the *normal projection* of f, and utilize P_f to study smoothness properties of f. It is first observed that for each $x \varepsilon D_f$ there is a unique linear manifold L_x which is maximal relative to the property [$x \varepsilon L_x$ and $f \mid D_f \cap L_x$ is differentiable at x]. (Notice that the dimension of L_x is k if and only if the hyperplanes supporting f at $(x, f(x))$ have $n - k$ "degrees of freedom.") Use of P_f shows quite simply that if $0 \leq k \leq n$ and S_k is the set of all $x \varepsilon D_f$ for which dim $L_x \leq k$, then S_k is the union of countably many compact sets of finite k-dimensional Hausdorff measure. (For $k = n - 1 = 1$ this appears to follow from results of Durand [5]. That the $(k + 1)$-dimensional measure of S_k is zero was stated without proof by Favard [6; 228] and proved for $k = n - 1 = 1$ by Caratheodory [1; 83] and by Reidemeister [15].)

A theorem is proved concerning the dimensionality of certain subsets of C-collections, and is used to show that no open set in E^n can be filled by a non-trivial *continuous* C-collection. Also discussed (by means of upper semi-continuous collections on the sphere) are some questions concerning antipodal points of convex bodies.

As general references on upper semi-continuous collections we mention R. L. Moore [14] and G. T. Whyburn [17], on convex sets, Bonnesen and Fenchel [2]. We occasionally refer to a theorem on convex sets to justify a statement about convex functions without further reference to the obvious relationships between the two.

The interior, boundary, closure, and dimension of a set X will be denoted by Int X, BX, Cl X, and dim X. The empty set will be denoted by Λ and the origin (neutral element) of the vector space under consideration by ϕ. We use $\cap$, $\cup$, and $\sim$ for set-theoretic intersection, union, and relative difference, $+$ and $-$ being reserved for vector addition and subtraction. If F is a family of sets, then the union and intersection of all sets in F will be denoted by σF and πF respectively. $\{x \mid P(x)\}$ will denote the set of all x for which $P(x)$ is true. "Upper semi-continuous" and "upper semi-continuity" will be abbreviated to u.s.c.

Received December 3, 1951; presented to the American Mathematical Society, December 28, 1951. Research of the first author was supported by the Office of Naval Research; the second author is a National Research Fellow.