ISOGONAL POINTS FOR A TETRAHEDRON

By N. A. Court

1. **Notations.** Let M, M' be a pair of isogonal conjugate points with respect to a tetrahedron $(T) = ABCD$; P, P'; Q, Q'; R, R'; S, S', the projections of M, M' upon the faces DBC, DCA, DAB, ABC of (T).

The tetrahedrons $(M) = PQRS$, $(M') = P'Q'R'S'$ are the pedal tetrahedrons of M, M' for (T); the common circumsphere (L) of (M), (M') is the common pedal sphere of M, M' for (T).

The lines $p = (QRS, Q'R'S')$, $q = (RSP, R'S'P')$, $r = \cdots$, $s = \cdots$, will be referred to as the *co-pedal lines* of M, M' for (T), or for the respective trihedrons of (T).

The spheres having A, B, C, D, for centers and orthogonal to (L) will be denoted by (A), (B), (C), (D).

2. **The pedal tetrahedrons.** (a) The points M, M' are isogonal for each of the four trihedrons of (T). Thus the planes PQR, $P'Q'R'$ and the sphere (L) are the pedal planes and the pedal sphere of M, M' for the trihedron of (T) having D for vertex.

The plane DBC cuts the sphere (D) along a great circle (d_a) and the sphere (L) along a small circle (l_a) orthogonal to (d_a). The points P, P' lie on the circle (l_a) whose center L_a is the projection of L upon the plane DBC. Now the center L of (L) is the mid-point of the segment MM' [1; 243, §747], hence L_a is the mid-point of the segment PP', that is, P, P' are diametrically opposite points on the circle (l_a). Consequently P, P' are conjugate points with respect to the great circle (d_a) and therefore also for the sphere (D).

The polar plane of P for (D) is perpendicular to the line DP and therefore also to the plane DBC passing through DP; moreover, this polar plane passes through the conjugate P' of P, and therefore contains the perpendicular $P'M'$ at P' to the plane DBC. Thus the point P is conjugate to the point M' for the sphere (D).

Similar considerations applied to the faces DCA, DAB of the trihedron D show that M' is also conjugate to the points Q, R for the sphere (D). Hence the polar plane of M' for this sphere coincides with the plane PQR.

In a like mannner it may be shown that the point M is the pole of the plane $P'Q'R'$ for (D). Thus:

If M, M' are two isogonal points for a trihedron, the pedal planes of M, M' for the trihedron coincide with the polar planes of M', M for the sphere having for center the vertex of the trihedron and orthogonal to the pedal sphere of M, M' for the trihedron.

Received October 22, 1951; presented to the American Mathematical Society, November 23, 1951.