COMPOSITIONS INVOLVING TERNARY CUBICS

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This note deals with identities A, B between components of composition of ternary cubics expressible as determinants of 3^2-matrices, the elements being linear in indeterminates x, y, z over a commutative and associative ring R with a modulus I.

Notation. Let (α) denote a row matrix $(\alpha_1, \alpha_2, \alpha_3)$; $\mathbf{A}(\alpha)$ a matrix over R linear in (α); $a(\alpha)$ a cubic in (α), generally the determinant of $\mathbf{A}(\alpha)$; $\mathbf{A}^*(\alpha)$ the adjoint of $\mathbf{A}(\alpha)$. The unit matrix is not written.

Definition. $f(x)$ *permits composition* in R if $f(x) = a(\alpha)\phi(\beta,\gamma)$, the (x) being polynomials over R in (α), (β), (γ). (This definition includes that of invariance.)

The method used is outlined in [4] and implies the following.

Lemma. *If* $(x)\mathbf{A}(\alpha) = (\alpha)\mathbf{F}(x)$ *and if* $(x) = (\beta)\mathbf{A}^*(\alpha)$, *then* $f(x) = a(\alpha)\phi(\beta,\alpha)$.

In the general case $\phi(\beta,\alpha)$ is too complicated for explicit expression here and two special cases are chosen as illustrations.

1. The binary cubic.

$$\mathbf{F}(x) = \begin{vmatrix} ax + by, & cx + dy, & 0 \\ 0, & x, & y \\ y, & 0, & x \end{vmatrix}; \qquad f(x) = ax^3 + bx^2y + cxy^2 + dy^3;$$

$$\mathbf{F}(x,y,z) = \begin{vmatrix} ax + by, & cx + dy, & (ad - bc)z \\ az, & x, & y - bz \\ y, & dz, & x - cz \end{vmatrix};$$

$$f(x,y,z) = f(x) + ad(ad - bc)z^3 + a(bd + c^2)z^2x - 2acx^2z + d(ac + b^2)z^2y - 2bdzy^2 - (bc + 3ad)xyz;$$

$$b(\beta) = d^2\beta_1^3 + a^2\beta_2^3 + ad\beta_3^3 + (c^2 - 2bd)\beta_1^2\beta_2 + ab\beta_2^2\beta_3 + bd\beta_3^2\beta_1 + (b^2 - 2ac)\beta_1\beta_2^2 + ac\beta_2\beta_3^2 + cd\beta_3\beta_1^2 + (bc - 3ad)\beta_1\beta_2\beta_3 .$$

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