# A PROPERTY OF HAUSDORFF MEASURE

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1. A 1–1 transformation $T$ of $n$-dimensional Euclidean space onto itself is said to be bounded if there exist two real numbers $K$, $k$ with $K \geq k > 0$ such that for any two points $p_1$, $p_2$ of the space $k\rho(p_1, p_2) \leq \rho(T(p_1), T(p_2)) \leq K\rho(p_1, p_2)$, where $\rho$ denotes the Euclidean distance and $T(p_1)$ is the transform of $p_1$ under $T$.

A real valued function $h(x)$ of a real variable $x$, defined for $x \geq 0$, is called a fractional measure function if

(i) $h(x)$ is continuous and strictly increasing,

(ii) $x^n/h(x)$ is an increasing function of $x$,

(iii) $h(0) = 0$, $\lim_{x \to 0+} x^n/h(x) = 0$.

For a point set $A$, let $\mathfrak{S}(A, \delta)$ denote an enumerable family of open convex sets whose point set sum includes $A$ and which is such that each convex set of the family has diameter less than $\delta$. Let $d(A)$ denote the diameter of $A$. The Hausdorff measure of $A$ with respect to the measure function $h(x)$, denoted by h.m.A, is defined by [2] h.m.A $= \lim_{\delta \to 0}$ {lower bound over all $\mathfrak{S}(A, \delta)$ of $\sum_{\mathfrak{S}(A,\delta)} h(d)$}.

If two measure functions $h(x)$, $H(x)$ are such that

$$\varliminf_{x \to 0+} h(x)/H(x) = \varliminf_{x \to 0+} H(x)/h(x) = 0,$$

they are said to be incomparable.

The object of this paper is to establish the following two theorems.

Theorem 1. *For any two fractional measure functions $H(x)$, $h(x)$ and any two positive numbers $\alpha$, $\beta$ there exist two perfect point sets $A$, $B$ such that*

(i) H.m.A $= \alpha$, h.m.B $= \beta$,

(ii) *for any bounded transformation $T$,* H.m.$(T(A)B) =$ h.m.$(T(A)B) = 0$.

Theorem 2. *If the conditions of Theorem 1 hold and, in addition, the measure functions are incomparable, the sets $A$ and $B$ can be chosen to satisfy conditions* (i), (ii), *and the additional condition*

(iii) h.m.A $= \infty$, H.m.B $= \infty$.

As a corollary to Theorem 1 it follows that for any fractional measure function $h(x)$ there exist two closed sets $A$, $B$ of finite $h$ measure such that it is not possible to find decompositions $A = A_0 + \sum_1^\infty A_n$, $B = B_0 + \sum_1^\infty B_n$ with the properties that

(i) h.m.$A_0$ = h.m.$B_0 = 0$,

(ii) $A_n$ is a translation of $B_n$.

(See [5]; a large number of related results have also been established by Piccard [4].)

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