CONVERGENCE IN TOPOLOGY

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This paper is an extension and simplification of results of Birkhoff [2] and Tukey [6] on Moore-Smith convergence. It is intended as a comprehensive and reasonably self contained treatment.

The principal novelty of the treatment stems from our generalization of the notion of subsequence. We shall prove, for example, that a topological space is (bi) compact if and only if every (generalized) sequence has a convergent (generalized) subsequence. This theorem (as well as Theorems 10 and 20 of this paper) is not true if "cofinal subset" is used as "generalized subsequence".

There are two types of convergence for which suitable and convenient generalizations of subsequence are available; namely, convergence of phalanxes [6] and filters [4]. However, the Moore-Smith theory seems to be the only one which embraces as direct special cases such diverse processes as convergence of sequences and double sequences, computation of infinite unordered sums, and computation of integrals as limits over refinement of subdivision. It therefore seems desirable to improve the general theory.

Our simplification of earlier proofs results largely from systematic use of Cartesian products of directed sets—a by no means new device.

I. Directed sets and nets.

Our definitions, save that of subnet, essentially follow Tukey [6].

1. Definitions. A binary relation $>$ *directs* a set D if D is non-void and

(a) if m, n and $p \in D$, and if $m > n$ and $n > p$ then $m > p$,

(b) if $m \in D$ then $m > m$, and

(c) if $m \in D$ and $n \in D$ then for some $p \in D$, $p > m$ and $p > n$.

If $m > n$ we say "*m follows n*".

A *directed set* is a pair $D, >$ where $>$ directs D. (For example, if M is the family of finite sets of integers, then $M, \supset$ is a directed set.)

A *net* is a function S, together with a relation $>$ which directs the domain of S. If D is the domain of S, then $S, >$ *is on* D. If a set X contains the range of S then $S, >$ is a net *in* X. If $n \in D$, the value of S at n is denoted by either S_n or $S(n)$.

A net $S, >$ on D is *eventually in* X if there is an element $p \in D$ such that if $n \in D$ and $n > p$ then $S_n \in X$.

A net $T, >_1$ on E is a *subnet* of a net $S, >$ on D if there is a function N on E to D such that

(a′) if $p \in D$ then N is eventually in the set of all $q \in D$ for which $q > p$, and

(b′) $T = SN$; that is, for $i \in E$, $T_i = S_{N_i}$.

Received October 27, 1947; in revised form, February 6, 1950.