SUB-ADDITIVE FUNCTIONS

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CHAPTER I. Elementary results on sub-additive functions

1.1. Sub-additive functions and angular semi-modules: definitions and examples. This paper will be concerned with the properties of measurable, real-valued functions of a point p of E_n, Euclidean space of n dimensions, satisfying the inequality

$$(1.1.1) \qquad f(p + p') \leq f(p) + f(p')$$

for all points p, p' in the domain of definition, described below. Sometimes the coordinates of the points will be explicitly mentioned, and (1.1.1) will be written

$$(1.1.1') \qquad f(x_1 + x'_1, \cdots, x_n + x'_n) \leq f(x_1, \cdots, x_n) + f(x'_1, \cdots, x'_n).$$

DEFINITION 1.1.1. Such functions are called *sub-additive.*

If the inequality is reversed, the functions will be termed *super-additive.* Sub-additive functions in E_1 have been treated fully by Hille [10; Chapter VI].

On occasion the domain of definition of the functions considered will be the entire space of n dimensions; frequently, however, the functions will be defined only on Σ_n, a measurable sub-set of E_n. In all cases Σ_n will be a semi-module, *i.e.*, if $p, p' \in \Sigma_n$ then $p + p' \in \Sigma_n$, and usually Σ_n will be an angular semi-module, S_n, defined as follows.

DEFINITION 1.1.2. A semi-module is *angular* if it is an open set and has the origin as a limit point.

While it is clearly desirable that the domain of definition of a sub-additive function be a semi-module, it is not so evident why it should be an angular semi-module. It will appear that many of the local properties of the functions are lost if the functions are not defined in the neighborhood of the origin; as the theory is developed it will be easy to see in most cases whether the domain of definition must be an angular semi-module or whether a semi-module would suffice.

An important angular semi-module is that region of E_n where all the coordinates are positive—let it be called E_n^+. A less significant angular semi-module is E_n^-, the region where all coordinates are negative. It is not hard to show that the only angular semi-modules in one dimension are E_1^+, E_1^-, and E_1 itself [11; Theorem 2.6].

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