# THE ROOTS OF TRIGONOMETRIC INTEGRALS

By N. G. de Bruijn

1. **Introduction.** Concerning the roots of trigonometric integrals G. Pólya (see references at the end of the paper) has proved a number of results which he derived from properties of the roots of polynomials. He proved, for instance, the reality of all the roots of the following functions:

$$\int_{-\infty}^{\infty} e^{-t^{2n}} e^{izt}\, dt \qquad (n = 1, 2, 3, \cdots); \tag{1.1}$$

$$\int_{-\infty}^{\infty} C(t) e^{izt}\, dt \qquad (\lambda > 0), \tag{1.2}$$

where $C(t) = \exp(-\lambda \cosh t)$, and

$$\int_{-\infty}^{\infty} \exp(-at^{4n} + bt^{2n} + ct^2) \exp izt\, dt, \tag{1.3}$$

where $a > 0$, $b$ real, $c \geq 0$, $n = 1, 2, 3, \cdots$. (See concerning (1.1), [7], [8]; concerning (1.2), [6], [8]; concerning (1.3), [8].)

Another important result of Pólya is the following one (see [8]): Suppose that the function $F(t)$ of the real variable $t$ satisfies

$$\begin{aligned} &F(t) \textit{ integrable over } -\infty < t < \infty;\ F(t) = (F(-t))^*, \quad -\infty < t < \infty; \\ &F(t) = O(e^{-|t|^b}) \textit{ for } t \to \pm\infty,\ b > 2. \end{aligned} \tag{1.4}$$

(The * indicates the conjugate imaginary.)

Let $\varphi(t)$ be an integral function of genus 0 or 1, with real roots only, and let the number $\gamma$ be $\geq 0$. If the function $F(t)$ is such that all the roots of the integral

$$\int_{-\infty}^{\infty} F(t) e^{izt}\, dt \tag{1.5}$$

are real, then the same holds for the function $\int_{-\infty}^{\infty} F(t)\varphi(it)e^{\gamma t^2} e^{izt}\, dt$.

The function $\varphi(it)e^{\gamma t^2}$ is easily seen to be the limit of a sequence of polynomials, all of whose roots are purely imaginary. Pólya's result, stated in other words, is that these functions are *universal factors*, which conserve the reality of the roots of any trigonometric integral whose integrand satisfies (1.4). Pólya also proved that the functions $\varphi(it)e^{\gamma t^2}$ indicated above are the only analytical functions with this property. The latter result will not be used in the present paper.

Received July 16, 1948.