RAMANUJAN SUMS AND THE AVERAGE VALUE OF ARITHMETIC FUNCTIONS

By Richard Bellman

1. **Introduction.** Let us consider the number-theoretic function $d(n)$, the number of divisors of n. Although this function possesses a quite erratic behavior, assuming its minimum value 2 at the primes, and satisfying the inequality $d(n) \geq \exp((1 - \epsilon) \log 2 \log n / \log \log n)$ over another infinite sequence of integers, it was shown by Dirichlet that $d(n)$ possesses an average value which is quite smooth. It is moreover easy to show that $d(n^k)$ and $d(n)^k$ for $k = 1, 2, \cdots$ have mean values which are powers of $\log n$. Hence, it might be expected that for many sequences $\{a_n\}$, which behave like polynomial sequences, the relation $\sum_{n \leq N} d(a_n) \sim c_1 N (\log N)^{c_2}$, where c_1 and c_2 depend upon the sequence $\{a_n\}$, would hold. That this cannot be true for sequences that increase too rapidly is clearly seen by considering the sequence $\{2^n\}$.

Using a device applicable only to the quadratic case, it was shown by Bellman and Shapiro [2] that $\sum_{n \leq N} d(an^2 + bn + c) \sim c_3 N \log N$, if the polynomial is irreducible, $c_3 = c_3(a, b, c)$, with $\log N$ replaced by $\log^2 N$ otherwise. The result is not elementary, although not difficult. For polynomials of higher degree, very little is known. It was shown by van der Corput [13] that $\sum_{n \leq N} d^l(f(n)) = O(N(\log N)^{c_4})$, where c_4 depends upon the exponent l and the polynomial $f(x)$. For sums of the type $\sum_{n \leq N} d(ab^n + c)$ nothing is known.

The present paper grew out of an attempt to evaluate $\sum_{n \leq N} d(f(n))$. Unfortunately, this sum barely but thoroughly escapes the method given below, which was outlined in [1]. Using the method we can, however, demonstrate

Theorem 1. *Let the number-theoretic function $\sigma_{-s}(n)$ be defined by*

$$\sigma_{-s}(n) = \sum_{k \mid n} k^{-s}. \tag{1.1}$$

Then, for $\operatorname{Re}(s) > 0$,

$$\sum_{n \leq N} \sigma_{-s}(f(n)) \sim c_5(s) N \tag{1.2}$$

for any integer-valued polynomial $f(x)$. Further $\sum_p \sigma_{-s}(f(p)) \sim c_6(s) N / \log N$ where the summation is over the primes less than or equal to N.

The result is only non-trivial for $0 < \operatorname{Re}(s) \leq 1$. The form of the constants will be explicitly determined subsequently. It will be seen that each becomes infinite for $s = 0$. Since $\sigma_0(n) = d(n)$, it is clear that if, in place of (1.2), we could obtain an equality with a sufficiently small error term and if we knew

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