THE PROJECTIVE SPACE

By Karl Menger

Introduction. The most striking feature of projective geometry is the principle of duality: If in a proposition about the projective space (the projective plane) we interchange points and planes (points and lines) we obtain a valid proposition. It thus seems natural to ask for a self-dual foundation of the theories in the sense that for every postulate the above interchange leads to another postulate, if not to the same proposition.

Some of the classical foundations lack self-duality both with regard to the primitive concepts and the postulates; *e.g.*, points are primitive concepts (*i.e.*, remain undefined) whereas planes are defined as particular sets of points. (Veblen and Young [6; 17] define "If P, Q, R are three points not on the same line, and l is a line joining Q and R, the class of all points on the lines joining P and the points of l is called the plane determined by P and l.") That every line is on at least three points is a postulate (*i.e.*, remains unproved), whereas the proposition that every line is on at least three planes (every point is on at least three lines) is proved from the postulates. One could, of course, obtain a self-dual set by adjoining to the system whatever dual propositions are missing; but it is equally clear that if we had a sufficient foundation before, we should have a redundant set of postulates after, the adjunction.

The author's foundation of projective geometry on an algebra of "flats" which can be "joined" and "intersected" seems to have been the first attempt to set up a self-dual foundation for projective geometry. (See [2]. In that paper, projective geometry was developed in conjunction with Boolean algebra, and related theories. See also [4] and preceding notes quoted in that paper on page 481. We refer the reader also to the lecture [1].) The algebraic method has since become widely known under the names of lattice theory, theory of structures, Theorie der Verbaende. In terms of the two basic operations we defined a part relation, corresponding to the classical incidence relation. We then revived Euclid's definition of a point as that which has no part (except itself and the vacuum), and we introduced the dual concept of a hyperplane as that which is not a part (except of itself and the universe).

Since one of the main features of the algebra of projective geometry is the synthesis of geometries of spaces of different dimensions, there seems to be still room for a self-dual development of the geometry of the three-dimensional space (the plane) in terms of points, lines, and planes (points and lines) and the incidence relation. The content of the present paper is such a theory based on ten (five) independent postulates. (A similar theory of the plane, which,

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