POWERFREE INTEGERS REPRESENTED BY LINEAR FORMS

By Harold N. Shapiro

1. **Introduction.** The distribution of the square-free integers has been investigated in the past in great detail. If we denote by $Q(x)$ the number of square-free integers $\leq x$, then it is well known [2; §18.6] that

$$Q(x) = 6\pi^{-2}x + O(x^{\frac{1}{2}}). \tag{1.1}$$

This has in fact been improved to (see [3])

$$Q(x) = 6\pi^{-2}x + o(x^{\frac{1}{2}}). \tag{1.2}$$

However, whereas (1.1) is quite elementary, (1.2) seems to be as deep as the Prime Number Theorem. The result (1.1) has been extended in many directions. For example, it is known [4; 633–637] that for $Q(k, l, x) =$ the number of square-free integers $\leq x$ which are congruent to l modulo k, we have

$$Q(k, l, x) = c_1 x + O(x^{\frac{1}{2}}), \tag{1.3}$$

where

$$c_1 = 6\pi^{-2}k^{-1} \prod_{\substack{p \mid (k,l) \\ p \nmid [k/(k,l)]}} (1 - p^{-1}) \Big[\prod_{p \mid k} (1 - p^{-2}) \Big]^{-1}.$$

The ease with which results such as (1.1) and (1.3) are obtained is to be greatly contrasted with the difficulty met with in proving the analogous theorems for primes. This suggests that it is perhaps possible to solve the analogue for square-frees of many of the still unsolved problems concerning the distribution of primes. For example, the question of whether or not there are infinitely many x for which x and $x + 2$ are both prime is still unsettled. Clearly then the corresponding problem for any finite set of linear forms $a_i x + b_i$, $i = 1, \cdots, k$, is at present equally hopeless. In passing it might be mentioned that a reasonable set of necessary conditions has been worked out for this general case (see [1]).

On the other hand, if we consider the linear forms $f_i(x) = a_i x + b_i$, $i = 1, \cdots, k$, and ask if there are infinitely many x such that the $f_i(x)$ are simultaneously square-free, a complete answer can be given. More precisely if we let $Q_2(N, f_1, \cdots, f_k)$ denote the number of $x \leq N$ for which the $f_i(x)$ are simultaneously square-free, then, subject to certain necessary restrictions, it can be shown that

$$Q_2(N, f_1, \cdots, f_k) = c_2 N + O(N^{2/3+\epsilon}), \tag{1.4}$$

where $c_2 = c_2(f_1, \cdots, f_k) > 0$ and ϵ is an arbitrarily small positive number.

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