THE RELATIONS BETWEEN SOLUTIONS OF THE DIFFERENTIAL EQUATION OF THE SECOND ORDER WITH FOUR REGULAR SINGULAR POINTS

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Introduction. The differential equation of the second order with four regular singular points has not been seriously studied, nor has a method been given of extending its integrals from one domain of existence to another. It is the object of this paper to connect the integrals determined locally by finding the relations between them where they are simultaneously defined.

The relations between the solutions determine the generators of the group of the differential equation, a group with three generators. The coefficients of the elements of the group depend on the solution of a particular difference equation which is not of a simple character. This difference equation is studied first. It then becomes possible to establish a contour integral which expresses a solution of the differential equation in a larger domain than the power series of the standard forms of Fuchs and thence deduce the relations between these forms in their common regions of existence.

I. **Integration of the Differential Equation.** The linear homogeneous differential equation of the second order with four regular singular points may be written in the form

$$\begin{aligned} z(z-1)(z-a)\frac{d^2y}{dz^2} &+ [(\alpha+\beta+1)z^2 - \{\alpha+\beta-\delta+1+(\gamma+\delta)a\}z \\ &+ a\gamma]\frac{dy}{dz} + \alpha\beta(z-q)y = 0. \end{aligned} \tag{1}$$

This equation is referred to as Heun's equation [4].

The singular points are $z = 0, 1, a, \infty$, the indices at these singularities being, respectively, $(0, 1-\gamma)$, $(0, 1-\delta)$, $(0, 1-\epsilon)$, (α, β) where ϵ is defined by the equation $\gamma + \delta + \epsilon = \alpha + \beta + 1$.

Heun's equation contains an arbitrary parameter q; for the only linear differential equation of order higher than the first which is made completely determinate by the assignment of its singularities and of the indices belonging to these singularities is an equation of the second order which, if it has ∞ for a singularity, has two other singularities in the finite part of the plane.

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